Välkomna till TSRT15 Reglerteknik
Föreläsning 12

- Summary of lecture 12
- Feedback from estimated states
- Integral action
- Review of course
Summary of lecture 11

States contain all information needed to predict the future of the system. I should thus contain all information needed to control the system

\[
  u(t) = -l_1 x_1(t) - l_2 x_2(t) - \ldots - l_n x_n(t) + L r(t)
  = -L x(t) + L r(t)
\]

Closed-loop system

\[
  \begin{align*}
    \dot{x}(t) & = A x(t) + B u(t) \\
    y(t) & = C x(t)
  \end{align*}
\]

\[
  \begin{align*}
    \dot{x}(t) & = (A - BL)x(t) + BL r(t) \\
    y(t) & = C x(t)
  \end{align*}
\]

The feedback vector is computed by placing the poles (compute \( L \) so that \( A-BL \) has desired eigenvalues: \( \text{det}(sI-(A-BL)) = \text{desired pole polynomial} \))

Alternatively, compute an \( L \) which minimizes a function which describes our compromise between good control and small inputs (LQ-control)

\[
  \int_0^\infty (x^T(t)Q x(t)x(t) + u^2(t))dt
\]
In reality, the whole state \( x(t) \) can almost never be measured.

To estimate (reconstruct) the state \( x(t) \) from measurements \( y(t) \), a simulation model is driven by the same input as the true system. Measurements that don’t match the output from the simulation are used to correct the state estimate via a gain \( K \).

\[
\dot{x}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t))
\]

The design is called an observer.

The behavior of the observer is designed by placing the eigenvalues for \( A - KC \) (estimation speed vs. sensitivity to noise \( \eta(t) \))

\[
e(t) = x(t) - \hat{x}(t), \quad \dot{e}(t) = (A - KC)e(t) + K\eta(t)
\]
Before we start with the main content of todays lecture, we briefly return to state feedback design

Fråga: How do we achieve integral action in the state feedback?

Up until now, we have designed $L_r$ to avoid stationary control error in step responses. This is however not enough if there are disturbances or model errors. Integral action is required

Consider a constant disturbance $v(t)$ acting on the system

$$
\dot{x}(t) = Ax(t) + Bu(t) + Fv(t)
$$

$$
y(t) = Cx(t)
$$

How can we extend the state feedback to guarantee $y(t) \rightarrow r(t)$ despite $v(t)$?
Integral action in state feedback

Introduce an integrator state

\[ \dot{x}_{n+1}(t) = r(t) - y(t) \]

Complete state-space model

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_{n+1}(t)
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_{n+1}(t)
\end{bmatrix}
+ 
\begin{bmatrix}
B \\
0
\end{bmatrix}
\begin{bmatrix}
u(t) \\
v(t)
\end{bmatrix}
+ 
\begin{bmatrix}
F \\
0
\end{bmatrix}
\begin{bmatrix}
u(t) \\
r(t)
\end{bmatrix}
\]

\[ y(t) = 
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_{n+1}(t)
\end{bmatrix}
\]

Feedback the extended state vector

\[ u(t) = -Lx(t) - l_{n+1}x_{n+1}(t) \]

If stable and \( v(t) \) and \( r(t) \) are constant, in stationarity we have \( -Cx(t) + r(t) = 0 \) according to the second row in the extended state-space model.
Feedback from estimated states

**Question:** We have designed a controller based on the states

\[ u(t) = -l_1 x_1(t) - l_2 x_2(t) - \ldots - l_n x_n(t) + L_r r(t) \]
\[ = -L \dot{x}(t) + L_r r(t) \]

What happens if we use the estimated states instead?

\[ u(t) = -L \hat{x}(t) + L_r r(t) \]

Is the system still stable? What happens with performance?
Feedback from estimated states

\[ r(t) \xrightarrow{L_r} u(t) \quad \text{Observer} \]

\[ \hat{x}(t) = Ax + Bu + K(y - C\hat{x}) \]

\[ \dot{x} = Ax + Bu \]

\[ y = Cx + Du \]
Feedback from estimated states

We thus have two coupled systems (the real and our simulation)

\[
\begin{align*}
\dot{x}(t) & = Ax(t) + B(-L\hat{x}(t) + L_rr(t)) \\
\dot{\hat{x}}(t) & = A\hat{x}(t) + B(-L\hat{x}(t) + L_rr(t)) + K(Cx(t) - C\hat{x}(t)) \\
y(t) & = Cx(t)
\end{align*}
\]

This is a linear system with 2n states (the system state and the observer state). We can use standard eigenvalue analysis to study stability.

It turns out though that it is easier if we perform a change of coordinates and work with the estimation error instead of the estimate

\[
e(t) = x(t) - \hat{x}(t) \Rightarrow \hat{x}(t) = x(t) - e(t)
\]
Feedback from estimated states

Coupled equations in the new variables

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B(-L(x(t) - e(t)) + L_rr(t)) \\
\dot{e}(t) &= (A - KC)e(t) \\
y(t) &= Cx(t)
\end{align*}
\]

In matrix form

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}(t)
\end{bmatrix} =
\begin{bmatrix}
A - BL & BL \\
0 & A - KC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} L_rr(t)
\]

\[
y(t) =
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix}
\]

Stable if **A-BL** and **A-KC** stable!

**Feedback and observer can be designed separately**
Feedback from estimated states

The estimation error $e(t)$ is obviously not controllable (driven only by itself)

This shows up in the form that the transfer function $r(t)$ to $y(t)$ only has $n$ poles (the uncontrollable poles in the observer disappears)

$$
Y(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI - \begin{bmatrix} (A - BL) & BL \\ 0 & (A - KC) \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} L_r R(s)
= \ldots
= C(sI - (A - BL))^{-1} BL_r R(s)
$$

Same as without an observer!

To good to be true? What we don’t see is transient behavior due to non-zero initial error in the estimation error
**Question**: Can our feedback from estimated states be interpreted as a standard feedback from measurements \(y(t)\) and feedforward from \(r(t)\)?

\[
U(s) = F_r(s)R(s) - F_y(s)Y(s)
\]

The controller is defined by the following equations

\[
\begin{align*}
\dot{x}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \\
u(t) &= -L\hat{x}(t) + L_r r(t)
\end{align*}
\]

The controller is thus a linear system with two inputs \((r(t)\) and \(y(t)\)) and one output \((u(t))\)
Feedback from estimated states

Laplace transform gives

\[
s\hat{X}(s) = A\hat{X}(s) + B(-L\hat{X}(s) + L_r R(s)) + K(Y(s) - C\hat{X}(s))
\]

\[
U(s) = -L\hat{X}(s) + L_r R(s)
\]

Solve for the estimate

\[
\hat{X}(s) = (sI - (A - BL - KC'))^{-1}(BL_r R(s) + KY(s))
\]

We identify the following controller

\[
U(s) = F_r(s)R(s) - F_y(s)Y(s)
\]

\[
F_r(s) = (1 - L(sI - A + KC + BL)^{-1}B)L_r
\]

\[
F_y(s) = L(sI - A + KC + BL)^{-1}K
\]
Feedback from estimated states

Just as before, we can define sensitivity functions

\[ S(s) = \frac{1}{1 + F_y(s)G(s)} , \quad T(s) = \frac{F_y(s)G(s)}{1 + F_y(s)G(s)} \]

These can also be computed explicitly by looking at the transfer function from a disturbance \( v(t) \) to output \( y(t) = Cx + v(t) \) (definition of the sensitivity function)

\[
S(s) = 1 - (C(sI - A + BL)^{-1}B)(L(sI - A + KC)^{-1}K) \\
T(s) = 1 - S(s)
\]

Aha! Although the closed-loop systems does not depend on the observer gain \( K \), the sensitivity functions do. Hence \( K \) can influence robustness etc.
Advanced observer concepts

**Reduced observers**

If we can measure some states exactly, there is no sense in estimating them.

The computations are not as straightforward though...

**Kalman filter**

Placing the poles for the observer is not obvious...

A more advanced approach is to use an optimality concept similar to LQ-control. Find an observer which is optimal, given an assumption about the relative size of measurement noise and system disturbances. The solution turns out to be a linear feedback of the estimation error, i.e. a standard observer.
Feedback from estimated states

**Exempel:** Velocity control in helicopter

Let us play around in MATLAB and try different controllers, observers and models.
The course is about **feedback of linear systems**, which we have seen can be represented in various ways

\[
\frac{d^n}{dt^n}y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_n y(t) = b_0 \frac{d^m}{dt^m}u(t) + b_1 \frac{d^{m-1}}{dt^{m-1}}u(t) + \ldots + b_m u(t)
\]

\[\Downarrow\]

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

\[
y(t) = Cx(t) + Du(t)
\]

\[\Downarrow\]

\[
Y(s) = G(s)U(s)
\]
We have defined roots to the characteristic equation, poles in the transfer function, and eigenvalues of the A-matrix.

All these concepts are the same, they represent the parameters in the exponential functions in the homogenous (initial-value dependent) solution to the underlying differential equation.

\[ y(t) = C_1 e^{\lambda_1 t} + \ldots + C_n e^{\lambda_n t} \]
By using a feedback structure, we can change the behaviour of the original system

We started with a PID controller

\[ u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \dot{e}(t) \]

By selecting the gains the poles are moved in the closed-loop system, i.e. we adjust the solution to the differential equation

\( P \) controls the speed of the solution
\( I \) controls the final value (we want to have \( y(t) = r(t) \) stationary)
\( D \) controls oscillations (used to reduce the oscillations)

Suitable choices of the gains are obtained by analyzing the position of the poles as a function of the gains
Course review

**Root-locus** is one analysis method

Root-loci do not give an answer to exactly how to select the gains, but help us to draw **general conclusions** (possible to stabilize, can complex poles be obtained, when does it become unstable, what happens for small gains,…)

![Root Locus](image)
A linear system is characterized by how much it amplifies and phase-shifts a sinusoidal input

\[ G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots (1 + \frac{s}{z_m})}{s^p(1 + \frac{s}{p_1}) \cdots (1 + \frac{s}{p_n})} \]

\[ \log(|G(i\omega)|) = \log(K) - p \log(\omega) + \sum_{i=1}^{m} \log(|1 + \frac{i\omega}{z_i}|) - \sum_{i=1}^{n} \log(|1 + \frac{i\omega}{p_i}|) \]

\[ \arg G(i\omega) = -\frac{p\pi}{2} + \sum_{i=1}^{m} \arctan\left(\frac{\omega}{z_i}\right) - \sum_{i=1}^{n} \arctan\left(\frac{\omega}{p_i}\right) \]
The logarithmic amplitude function is almost piecewise linear in log-log scale. The curve **bends down when a pole is passed and up when a zero is passed**

\[ G(s) = 0.01 \frac{(1 + \frac{s}{1})(1 + \frac{s}{100})}{s(1 + \frac{s}{10})(1 + \frac{s}{1000})} \]
System properties can be seen in step-responses, pole positions, Bode plots and transfer functions. **Important to understand connections**…

\[ G_1(s) = \frac{2^2}{(s^2 + 2 \cdot 0.2 \cdot 2s + 2^2)} \]
\[ G_2(s) = \frac{4^2}{(s^2 + 2 \cdot 0.2 \cdot 4s + 4^2)} \]

\[ \omega_0 = \frac{2\pi}{T} \]

\[ \approx \omega_0 \]
The Bode plot also reveals what happens when the loop is closed with a P-controller with $K=1$

Stability requires $\arg(G(i)) > -180^\circ$, $|G(i)| = 1$

Stability requires $\arg(G(i)) > -180^\circ$
da $|G(i)| = 1$
Course review

By changing the P-controller to a more advanced controller, more phase can be achieved, and the crossover frequency can be changed.

The lead-link picks the desired crossover frequency and how much phase margin there should be there.

The lag-link is used to increase the gain in the frequency 0.
Control design is to a large degree about the compromise between small sensitivity function and small complementary sensitivity function.

\[ S(s) + T(s) = 1 \]
State-space model gave us an alternative view on linear systems

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Laplacetransformera

\[Y(s) = G(s)U(s)\]

Observable form
Controllable form

Stability analysis is done by checking the eigenvalues of \( A \), which turns out to be the poles of the transfer function

With a state-feedback \( u = -Lx \), the system behaviour can be changed by placing the poles of the closed-loop state-space matrix \( A-BL \)

The benefit with the state-space methodology is that it is suitable for computer aided design of controllers (such as LQ-control)
Course review

Observers gave us the possibility to implement state-feedback also when we cannot measure all states

\[
\dot{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \\
u(t) = -L\hat{x}(t) + L_tr(t)
\]

By analysing the complete system with observer and feedback from estimated states, we saw that it was nothing more than a normal feedback of the measurement \( y \).

\[
U(s) = F_r(s)R(s) - F_y(s)Y(s) \\
F_r(s) = (1 - L(sI - A + KC + BL)^{-1}B)L_r \\
F_y(s) = L(sI - A + KC + BL)^{-1}K
\]

State-feedback and observers do however give us an alternative, and in some cases, a more structured way to design the controller.
Course review

What is not part of the course

Null-space interpretation of controllability and observability (determinant criteria is enough)

Hall-, Nichols- och Nyquist plots (Bode is enough…)

Cascade structures and Smith-predictors (feedback and feedforward is enough)

Implementation
But wait, there’s more!

**Reglerteknik fortsättningskurs M**
Analysis of nonlinear systems, multivariable systems, optimal observers.

**Modellbygge och simulering**
Structured approach to modelling, more advanced model types, deriving models from measurement data.

**Industriell reglerteknik**
More advanced control structures and methods, implementation.

**Project course**
Often from companies (multivariable control of turbo engine, golf playing robot, autonomous boats/cars/aircrafts etc).
But wait, there’s more at ISY!

**Fordonsdynamik med reglering (at fordonssystem)**
Modelling and control of vehicles (primarily cars)

**Fordonssystem (på fordonssystem)**
Everything you need to know about modern cars...

**Konstruktion med mikrodatorer (på datorteknik)**
Project course with design of stuff controlled with micro computers (emphasis on the micro computer aspects, not a control course)
and IEL!

Mechatronics

Mechatronic systems project
Summary

Summary of todays lecture

- Integral action can be achieved, but it requires an extension of the state-space model with an extra state
- Feedback and observer can be designed separately without destroying stability properties for each other
- The closed-loop transfer function does not depend on the observer!
- The sensitivity function do however
- Pole palcement is tricky for observers, convenient to use Kalman filters in MATLAB to compute optimal placements
Summary

Important words

**Integrator state:** An artificial state in the controller added to obtain integral action in a state-feedback setting

**Reduced observer:** An observer which only estimates the states it cannot measure

**Kalman filter:** an observer designed based on a model of how large disturbances and measurements relatively to each other