Le 7: point-mass filter and particle filter

Whiteboard:
- Derivation and explanation of the PMF and PF.

Slides:
- Point-mass filter (PMF)
- Particle filter (PF)
- Examples and applications
Lecture 6: summary

Key tool for a unified derivation of KF, EKF, UKF.

\[
\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix} \right)
\]

\[\implies (X|Y = y) \sim \mathcal{N} \left( \mu_x + P_{xy} P_{yy}^{-1} (y - \mu_y), P_{xx} - P_{xy} P_{yy}^{-1} P_{yx} \right).\]

The Kalman gain is in this notation given by \( K_k = P_{xy} P_{yy}^{-1} \).

- In KF, \( P_{xy} \) and \( P_{yy} \) follow from a linear Gaussian model.
- In EKF, \( P_{xy} \) and \( P_{yy} \) can be computed from a linearized model (requires analytic gradients).
- In EKF and UKF, \( P_{xy} \) and \( P_{yy} \) computed by NLT for transformation of \((x^T, v^T)^T\) and \((x^T, e^T)^T\), respectively. No gradients required, just function evaluations.
Point-Mass Filter
Chapter 9 Overview

Particle filter

- Algorithms and derivation.
- Practical and theoretical issues.
- Computational aspects.
- Marginalization to beat the curse of dimensionality.
Model and Bayesian Recursion

General nonlinear state-space model:

\[ x_{k+1} = f(x_k, u_k, v_k) \]
\[ y_k = h(x_k, u_k, e_k) \]

or even more general Markov model:

\[ x_k \sim p(x_k|x_{k-1}) \]
\[ y_k \sim p(y_k|x_k) \]
Bayes Optimal Filter: measurement update

Bayes’ rule

\[ p(A|B, C) = \frac{p(B|A, C)p(A|C)}{p(B|C)}, \]

using \( A = x_k, (B, C) = y_{1:k}, B = y_k \) and \( C = y_{1:k-1} \), yields

\[ p(x_k|y_{1:k}) = \frac{p(y_k|x_k, y_{1:k-1})p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})} = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})}. \]

The Markov property \( p(y_k|x_k, y_{1:k-1}) = p(y_k|x_k) \) was used in the last equality.
Bayes Optimal Filter: time update

Bayes’ rule $p(A, B|C) = p(A|B, C)p(B|C)$ gives

$$p(x_{k+1}, x_k|y_{1:k}) = p(x_{k+1}|x_k, y_{1:k}) p(x_k|y_{1:k}) = p(x_{k+1}|x_k) p(x_k|y_{1:k}).$$

Again, the Markov property was used in the last equality. Marginalization of $x_k$ by integrating both sides with respect to $x_k$ yields

$$p(x_{k+1}|y_{1:k}) = \int p(x_{k+1}|x_k)p(x_k|y_{1:k}) \, dx_k.$$ 

This is known as the *Chapman-Kolmogorov* equation.
Bayes Optimal Filter: summary

General Bayesian recursion (time and measurement updates)

\[ p(x_{k+1}|y_{1:k}) = \int p(x_{k+1}|x_k)p(x_k|y_{1:k}) \, dx_k, \]

\[ p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})}. \]

- Analytic solution available in a few special cases: linear Gaussian model (KF) and finite state-space model (HMM).
- However, for a given trajectory \( x_{1:k} \), the recursion can be computed (neglect the integral).
- We can numerically evaluate different trajectories by comparing their likelihoods. But there are many possible trajectories, so Monte Carlo sampling of trajectories directly does not work.
Numerical Approximation

**Basic idea:** postulate a discrete approximation of the posterior. For the predictive density, we have

$$
\hat{p}(x_k|y_{1:k-1}) = \sum_{i=1}^{N} w^{(i)}_{k|k-1} \delta(x_k - x^{(i)}_k).
$$

The first moments (mean and covariance) are simple to compute from this approximation:

$$
\hat{x}_{k|k-1} = \mathbb{E}(x_k) = \sum_{i=1}^{N} w^{(i)}_{k|k-1} x^{(i)}_k,
$$

$$
P_{k|k-1} = \text{cov}(x_k) = \sum_{i=1}^{N} w^{(i)}_{k|k-1} (x^{(i)}_k - \hat{x}_{k|k-1})(x^{(i)}_k - \hat{x}_{k|k-1})^T.
$$

Also, the MAP estimate can be useful:

$$
\hat{x}^{\text{MAP}}_{k|k-1} = \arg \max_{x_k^{(i)}} \hat{p}(x_k|y_{1:k-1}).
$$
Measurement Update

The measurement follows directly, without any extra approximation

\[
\hat{p}(x_k|y_{1:k}) = \sum_{i=1}^{N} \frac{1}{c_k} p(y_k|x_k^{(i)}) w_k^{(i)}|_{k-1} \delta(x_k - x_k^{(i)}) \]

\[
c_k = \sum_{i=1}^{N} p(y_k|x_k^{(i)}) w_k^{(i)}|_{k-1}.
\]

The normalization constant \(c_k\) corresponds to assuring that \(\sum_{i=1}^{N} w_k^{(i)}|_{k} = 1\).
Time Update

Bayesian time update gives a continuous distribution

$$\hat{p}(x_{k+1}|y_{1:k}) = \sum_{i=1}^{N} w_{k|k}^{(i)} p(x_{k+1}|x_{k}^{(i)}).$$

To keep the approximation form, the distribution is sampled at points $x_{k+1}^{(i)}$, and the weights are updated as

$$w_{k+1|k}^{(i)} = \hat{p}(x_{k+1}^{(i)}|y_{1:k}) = \sum_{j=1}^{N} w_{k|k}^{(j)} p(x_{k+1}^{(i)}|x_{k}^{(j)}), \quad i = 1, 2, \ldots, N.$$  

There are two principles:

- Keep the same grid, so $x_{k+1}^{(i)} = x_{k}^{(i)}$, which yields the point mass filter.
- Generate new samples from the posterior distribution $x_{k+1}^{(i)} \sim \hat{p}(x_{k+1}|y_{1:k})$, which yields the marginal particle filter.

Both alternatives have quadratic complexity ($N$ weights $w_{k+1|k}^{(i)}$, each one involving a sum with $N$ terms).
Advantages:

- Simple to implement.
- Works excellent when $n_x \leq 2$.
- Gives the complete posterior, not only $\hat{x}$ and $P$.
- Global search, no local minima.

Problems:

- Grid inefficient in higher dimensions, since the probability to be at one grid point depends on the transition probability from all other grid points.
- The grid should be adaptive (rough initially, then finer).
- Sparse matrices can be used for multi-model distributions.
PMF Example: 2 DOA sensors, 2 targets

- Data from the FOCUS project (map overlay).
- Two microphone arrays (black x) compute two DOA.
- Two road-bound targets (green *).
- One grid point (stem plot) every meter on the road.

[Image of 3D plot with coordinates and grid points]
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http://youtu.be/VcdTebC9uTs
Particle Filter
Particle Filter

- Trick to avoid cubic complexity: sample trajectories, not states
- Time update for trajectory:

\[
p(x_{1:k+1}^{(i)} | y_{1:k}) = \underbrace{p(x_{k+1}^{(i)} | x_{1:k}^{(i)}, y_{1:k}) \cdot p(x_{1:k}^{(i)} | y_{1:k})}_{p(x_{k+1}^{(i)} | x_{k}^{(i)}) \cdot w_{k|k}^{(i)}}
\]

\[= w_{k|k}^{(i)} p(x_{k+1}^{(i)} | x_{k}^{(i)}) = w_{k+1|k}^{(i)}.\]

No sum involved here!

- The new sample is sampled from the prior in the original PF (SIR, or bootstrap, PF)

\[x_{k+1}^{(i)} \sim p(x_{k+1} | x_{k}^{(i)}).\]
Basic SIR PF Algorithm

Choose the number of particles $N$.

Initialization: Generate $x_0^{(i)} \sim p_{x_0}, i = 1, \ldots, N$ particles.

Iterate for $k = 1, 2, \ldots, t$:

1. **Measurement update**: For $k = 1, 2, \ldots,$

   $$w_k^{(i)} = w_{k-1}^{(i)} p(y_k | x_k^{(i)}).$$

2. **Normalize**: $w_k^{(i)} := w_k^{(i)} / \sum_j w_k^{(j)}.$

3. **Estimation**: MMSE $\hat{x}_k \approx \sum_{i=1}^N w_k^{(i)} x_k^{(i)}$ or MAP.

4. **Resampling**: Bayesian bootstrap: Take $N$ samples with replacement from the set $\{x_k^{(i)}\}_{i=1}^N$ where the probability to take sample $i$ is $w_k^{(i)}$. Let $w_k^{(i)} = 1/N$.

5. **Prediction**: Generate random process noise samples

   $$v_k^{(i)} \sim p_{v_k}, \quad x_{k+1}^{(i)} = f(x_k^{(i)}, v_k^{(i)}).$$
PF Code

Input arguments: NL object \( m \), SIG object \( z \).
Output arguments: SIG object \( z_{\text{hat}} \).

```matlab
y = z.y.;
u = z.u.;
xp = bsxfun(@plus, m.x0.', rand(m.px0, Np));  % Initialization
for k = 1:N
    % Time update
    v = rand(m.pv, Np);  % Random process noise
    xp = m.f(k, xp.', u(:,k), m.th).'; + v;  % State prediction
    % Measurement update
    yp = m.h(k, xp.', u(:, k).', m.th).';  % Measurement prediction
    w = pdf(m.pe, bsxfun(@minus, y(:,k).', yp));  % Likelihood
    xhat(k,:) = mean(bsxfun(@times, w(:,), xp));  % Estimation
    [xp, w] = resample(xp, w);  % Resampling
    xMC(:, k, :) = xp;  % MC uncertainty repr.
end
```
Example: 1D terrain navigation

Problem: Measured velocity $u_k$, unknown velocity disturbance $v_k$, known altitude profile $h(x)$ which is observed with $y_k$.

Model:

$$x_{k+1} = x_k + u_k + v_k,$$
$$y_k = h(x_k) + e_k,$$

Step 1. MUP $p(x_1 | y_1)$

Step 4. Resampling $p(x_1 | y_1)$

Step 5. TUP $p(x_2 | y_1)$
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http://youtu.be/thNh0E6tmV0
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Same assumptions as in 1D: aircraft measures ground altitude as measurement $y_k$ and noisy speed $u_k = v_k + w_k$, terrain elevation map (TAM) provides $h(x_k)$.

Dynamic model

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\[ y_k = h(x_k) + e_k \]
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Wheel speed sensors provide speed and yaw rate, street map provides constraints on turns.

Video illustrates how a uniform prior over a part of the road network eventually converge to a single particle cluster when sufficient information is obtained (but many local particle clusters initially).
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WiMAX network (Brussels here) provides signal strength measurements, RSS map provides a fingerprint $h(x_k)$, street map (optional) provides constraints on turns.

First half of video shows PF without street constraint, second half with road constraint (with superior performance).
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Underwater vessel measures its own depth and distance to bottom, and sea chart provides depth $h(x_k)$.

Video shows how a uniform prior quickly converges to a unimodal particle cloud. Note how the cloud changes form when passing the ridge.
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On-board radar measures range to shore and possible its speed, sea chart provides conditional distance to shore $h(x_k)$.

First video shows animated ship and radar measurements. Second video shows radar measurements overlayed on sea chart (given estimated position), the estimated (PF) position of the own ship, and the estimated (EKF) positions of other ships and 1 minute prediction of their position (collision warning).
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Summary
Lecture 7: summary

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   $$
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   $$
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