Reading instructions

- The names and numbers of the chapters in this exercise collection are consistent with the names and numbers of the chapters in the textbook.
- Starred (⋆) exercises deal with discrete-time systems and are optional.
1 Introduction

1.1

Consider the linear feedback control system given by the figure below.

\[ \Sigma \rightarrow G(s) \rightarrow -F(s) \]

Show that if the small gain theorem (Swe: lägförstärkningssatsen) is fulfilled the Nyquist criterion is also fulfilled.

1.2

Consider a static nonlinear system described by an ideal relay given by the function

\[ y(t) = f(u(t)) = \begin{cases} 
1, & u > 0 \\
0, & u = 0 \\
-1, & u < 0 
\end{cases} \]

What is the gain of the relay?
1.3

Consider the system

\[ Y(s) = G(s)U(s) \quad G(s) = \frac{2}{s^2 + 2s + 2} \]

The control signal goes through a valve with saturation

\[ \tilde{u}(t) = \begin{cases} 
1, & \text{if } u(t) > 2 \\
\frac{1}{2}u(t), & \text{if } |u(t)| \leq 2 \\
-1, & \text{if } u(t) < -2 
\end{cases} \]

The output is thus

\[ y(t) = G(p)\tilde{u}(t). \]

The system is controlled using proportional feedback, i.e. \( u(t) = -Ky(t) \). For what values of \( K \) is the closed-loop system guaranteed to be stable according to the small gain theorem?

1.4

Compute the norms \( \| \cdot \|_\infty \) and \( \| \cdot \|_2 \) of the continuous-time signals

(a) \[ y(t) = \begin{cases} 
a \sin(t), & t > 0 \\
0, & t \leq 0 \end{cases} \]

(b) \[ y(t) = \begin{cases} 
\frac{1}{t}, & t > 1 \\
0, & t \leq 1 \end{cases} \]

(c) \[ y(t) = \begin{cases} 
e^{-t}(1 - e^{-t}), & t > 0 \\
0, & t \leq 0 \end{cases} \]
1.5

Consider the linear system

\[ G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}. \]

Compute the system gain \( \|G\| \) for all values of \( \omega_0 > 0 \) and \( \zeta > 0 \).

1.6

Analyze the stability of the following system, first by using the small gain theorem and then by computing the poles of the closed-loop system. Explain possible differences.

\[ K \frac{a}{s + a} \]

1.7

Consider the feedback control system

where \( G(s) \) is a linear system with the magnitude plot
and \( f(\cdot) \) is an amplifier with the following input-output relationship

Is the closed-loop system stable?

1.8

Consider a DC motor given on state-space form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_2 + au \\
y &= x_1
\end{align*}
\]

The inverse time constant \( a \) can vary as

\[
a = 1 + \rho, \quad |\rho| < \delta.
\]

The system is controlled using a proportional controller \( u(t) = -Ky(t) \).

Suppose that \( a \) is constant. Give a sufficient condition on \( K \) such that the closed-loop system is stable for all \( a \).
Once again consider the DC motor in exercise 1.8 but now assume that the parameter $a$ can vary arbitrarily fast with time

$$a = a(t) = 1 + \rho(t), \quad |\rho(t)| < \delta, \quad \forall t$$

(a) Introduce a new, artificial input signal $w$ and a new artificial output signal $z$ such that the system can be described by the feedback connection below

(b) Consider the time-varying, static system from $z(t)$ to $w(t)$:

$$w(t) = \rho(t) \cdot z(t), \quad |\rho(t)| < \delta, \quad \forall t.$$  

Show that the gain of this system (according to Definition 1.1 in the textbook) is at most $\delta$.

(c) Give a sufficient condition on $K$, for instance an inequality that implicitly characterizes $K$, for the closed-loop system to be stable no matter how $a(t)$ varies with time.
2 Representation of Linear Systems

2.1

A simplified model of an alternating-current generator can be described as follows. The input signals to the system are the magnetizing current $I_m$, which is fed into the armature winding, and the driving torque $M$ which is applied to the rotor axis. The rotation speed of the generator is $\omega$, and the change in rotation speed is given by

$$J \dot{\omega} = M - M_e$$

where

$$M_e = K_e \cdot \omega \cdot I_f$$

is the electrical torque due to the emf. $I_f$ is the current in the stator winding, given by the relationship

$$e = R \cdot I_f$$

where the voltage $e$ is generated in the stator winding according to

$$e = C_e \cdot I_m \cdot \omega$$

and $R$ is the load resistance applied to the stator winding. Consider $e$ and $\omega$ as output signals, $M, I_m$ and $R$ as input signals. Set $K_e = C_e = J = 1$ and find a state-space representation for this system.

Linearize around the stationary point

$$\omega_0 = R_0 = I_{m0} = M_0 = 1$$

and derive the transfer function matrix from

$$u = \begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix} \text{ to } y = \begin{bmatrix} \Delta \omega \\ \Delta e \end{bmatrix}$$
Consider the system consisting of two coupled tanks described in the figure below.

The flow of water into the left and right halves of the tank are denoted \( u_1 \) and \( u_2 \) respectively. These flows are the input signals. The water levels in the two halves are denoted \( h_1 \) and \( h_2 \) respectively. The flow \( y \) out from the tank is assumed to be proportional to the water level in the right half of the tank

\[
y(t) = \alpha h_2(t)
\]

The flow between the two halves is proportional to the difference between the levels

\[
f(t) = \beta (h_1(t) - h_2(t))
\]

where a flow from left to right is considered positive. Let \( h_i, u_i \) and \( y \) be deviations from nominal values. Thus, they can have negative values. Assume that the area of the halves are \( A_1 = A_2 = 1 \).

(a) Derive the transfer function from \( u_1, u_2 \) to \( y \).

(b) Compute the maximum and minimum singular values of \( G(0) \) and give an intuitive explanation to the corresponding input signals.
2.3

Find a state-space realization of the system

\[ G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix} \]

2.4

Find a state-space realization of the system

\[ y(t) = \frac{p}{p^2 + 4p + 4} u_1(t) + \frac{p - 1}{p^2 + 5p + 6} u_2(t) \]

2.5

A system is described by the differential equation

\[ \ddot{y} + a_1 \dot{y} + a_2 y = b_{11} \dot{u}_1 + b_{12} u_1 + b_{21} \dot{u}_2 + b_{22} u_2. \]

Find a state-space realization.

2.6

Consider the system

\[
\begin{align*}
\dot{y}_1 + y_2 &= \dot{u} + 2u \\
\dot{y}_2 + y_2 + y_1 &= u
\end{align*}
\]

Find a state-space realization.
3 Properties of Linear Systems

3.1
Consider the transfer function matrix

\[ G(s) = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} & \frac{1}{s+2} \end{pmatrix} \]

Derive the pole and the zero polynomials of the system? What is the dimension of a minimal state-space realization?

3.2
Find the poles and zeros of

\[ G(s) = \frac{1}{(s+1)(s+3)} \begin{pmatrix} 1 & 0 \\ -1 & 2(s+1)^2 \end{pmatrix} \]

3.3
Find the poles of

\[ G(s) = \frac{1}{(s+1)^2} \begin{pmatrix} 1-s & \frac{1}{3} - s \\ 2 - s & 1 - s \end{pmatrix} \]

What is the dimension of a minimal realization?

3.4
(a) Consider the system

\[ G(s) = \begin{pmatrix} \frac{s+5}{s^2+3s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{pmatrix} \]

What is the dimension of a minimal state-space realization?
(b) Consider the system

\[ G(s) = \begin{pmatrix} \frac{s+5}{s^2+3s+2} & \frac{1}{s+2} \\ \frac{1}{s+4} & \frac{1}{s+4} \end{pmatrix} \]

What is the dimension of a minimal state-space realization?

3.5

A system has the following input-output relation

\[
\begin{align*}
\dot{y}_1 + y_1 - \dot{y}_2 &= u_1 - u_2 \\
\dot{y}_2 + \dot{y}_1 + y_2 &= u_1 + u_2.
\end{align*}
\]

Find a matrix fraction description, \( y(t) = A(p)^{-1}B(p)u(t) \), and compute the poles and zeros of the system.

3.6

Consider the MIMO system:

\[
\begin{align*}
\dot{x}(t) &= \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t)
\end{align*}
\]

Find a minimal realization of the system, i.e. a realization that is controllable and observable.
Consider the multivariable system

\[ Y(s) = G(s)U(s) \]

where

\[ G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+3} & \frac{1}{s+4} \end{pmatrix} \]

(a) Determine the maximum and minimum singular value of the frequency response at the frequency \( \omega = 2 \) rad/s.

(b) Determine also the input vectors, in terms of their Fourier transforms, corresponding to the largest and smallest gain of the system at \( \omega = 2 \).

(c) Generate, in Matlab, an input vector that corresponds to the largest gain of the system and simulate the system using this input.

**Hint:** Use sinusoidal input signals and use the following properties of a sinusoidal signal considered over a finite time interval.

- The Fourier transform is proportional to the amplitude of the sinusoidal signal, i.e. for \( u_1(t) = A \sin \omega t \) the Fourier transform \( U_1(i\omega) \) is proportional to \( A \).
- Time delay of a signal corresponds to a change of the argument of the Fourier transform, i.e. if \( u_1(t) = A \sin \omega t \) has the transform \( U_1(i\omega) \) the signal \( A \sin(\omega t + \phi) \) has the transform \( U_1(i\omega)e^{i\phi} \).

(d) Verify that the obtained output signals correspond to largest gain of the system.

### 3.8

Find the poles and zeros of

\[ G(s) = \begin{pmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{s+2} & \frac{s-1}{s+2} \end{pmatrix} \].
5 Disturbance Models

5.1

A continuous-time stochastic process \( u(t) \) has the power spectrum \( \Phi_u(\omega) \). For the power spectra below, find linear filters such that the processes can be represented as white noise fed through those filters.

(a) \[ \Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2} \]

(b) \[ \Phi_u(\omega) = \frac{a^2b^2}{(\omega^2 + a^2)(\omega^2 + b^2)} \]

5.2

A position sensor is mounted on a machine that vibrates with a frequency around 5 Hz, and this causes that a disturbance \( n(t) \) affects the position measurement. In order to include the properties of the measurement disturbance in the control design one formulates a model that describes the properties of the disturbance as filtered white noise \( V \). The following models are suggested

(i) \( N(s) = \frac{1}{s + 0.001}V(s) \)

(ii) \( N(s) = \frac{900}{s^2 + 6s + 900}V(s) \)

(iii) \( N(s) = \frac{25}{s^2 + s + 25}V(s) \)

Which disturbance model is the best choice?
5.3

Consider a missile propelled by the thrust $u$. The missile’s position is $z$. A simplified model for the air drag is

$$f = k_1 \cdot \dot{z} + v$$

where $v$ are, more or less, random wind gusts.

(a) Derive a state-space representation and an input-output representation for how the controlled output $z$ depends on $u$ and $v$.

(b) The system disturbance $v$ has the spectral density

$$\Phi_v(\omega) = k_0 \cdot \frac{1}{\omega^2 + a^2}$$

Modify the state-space representation in (a) to make it possible to express the system disturbance using white noise. What is the corresponding transfer function?

5.4

Assume, in exercise 5.3, that the position $z$ is measured with an error

$$y(t) = z(t) + n(t)$$

Derive a state-space model for the missile if

(a)

$$\Phi_n(\omega) = 0.1$$

(b)

$$\Phi_n(\omega) = 0.1 \frac{\omega^2}{\omega^2 + b^2}.$$  

(c)

$$\Phi_n(\omega) = 0.1 \frac{1}{\omega^2 + b^2}.$$
5.5

A system has the state-space representation

\[ \dot{x} = Ax + Bu + Nw \]
\[ y = Cx + n \]

We assume that the system disturbance \( w \) changes stepwise and that the measurement noise is periodical with a frequency of about 2 Hz.

Modify the state-space representation to make it possible to model the disturbances.

5.6

In airplanes it is common to measure acceleration as well as speed. The acceleration is measured using accelerometers and the speed is calculated from measurements of air data, such as dynamical pressure et cetera. Thus, the measurements are independent, but of course they are related to each other.

(a) Derive a state-space model for the speed and acceleration. Let the measured speed and acceleration be output signals and assume that the derivative of the acceleration is white noise. Furthermore, assume that the measurement errors in speed and acceleration are white noises, independent of each other.

(b) Discuss how we can get better estimates of the speed and acceleration using Kalman filtering.

5.7

The depicted dynamical system is described by the differential equation
\[ \ddot{x}(t) + x(t) = v(t) \]

The external force \( v(t) \) is white noise with

\[
\begin{align*}
E v(t) &= 0 \\
E v(t)v(s) &= \delta(t - s)
\end{align*}
\]

We want to estimate the position \( x(t) \) and speed \( \dot{x}(t) \) at every time instant. We have sensors for both speed and position but for economical reasons we only want to use one sensor. We can choose between

**Alternative I:** The measured signal is

\[ y_1(t) = x(t) + e_1(t) \]

**Alternative II:** The measured signal is

\[ y_2(t) = \dot{x}(t) + e_2(t) \]

The measurement errors are \( e_1(t) \) and \( e_2(t) \). For simplicity we assume that they are both white noises with

\[
\begin{align*}
Ee_1(t) &= Ee_2(t) = E[e_1(t)e_2(s)] = 0 \\
Ee_1(t)e_1(s) &= Ee_2(t)e_2(s) = \delta(t - s)
\end{align*}
\]

For each alternative derive the linear filter that, in steady state, yields the best estimate of \( x(t) \) and \( \dot{x}(t) \), in the sense of smallest variance of the estimation error, from measurements up to and including time \( t \). State, with an explanation, which alternative you think is the best.
Consider the depicted radar antenna.

From noisy measurements of the position of the antenna $\Theta_m$ we want to estimate the true position $\Theta$. To be able to do this we need a model of the system. To this end, describe the dynamics of the antenna with

$$J\ddot{\Theta}(t) + B\dot{\Theta}(t) = \tau(t) + \tau_d(t),$$

where $J$ is the moment of inertia for the moving parts of the antenna, $B$ is the coefficient of viscous friction, $\tau(t)$ is the torque produced by the motor, and $\tau_d(t)$ is the torque caused by the wind. Assume that $\tau_d(t)$ can be modeled as white noise. Furthermore, assume that the torque $\tau(t)$ is proportional to the motor voltage, $\mu(t)$, i.e.

$$\tau(t) = k\mu(t)$$

Finally, let us for simplicity, assume that the measurement error can be modeled as additive white noise $e_m(t)$. Hence, the output signal is

$$\Theta_m(t) = \Theta(t) + e_m(t).$$

Discuss how $\Theta(t)$ can be estimated from $\Theta_m(t)$ using a Kalman filter.

Technical data:

\[
\begin{align*}
B/J &= 4.6 \text{ s}^{-1} \\
\frac{k}{J} &= 0.787 \text{ rad/ Vs}^2 \\
J &= 10 \text{ kg m}^2 \\
E \tau_d(t)\tau_d(s) &= v_d\delta(t - s) = 10 \text{ N}^2m^2 \cdot \delta(t - s) \\
E e_m(t)e_m(s) &= v_m\delta(t - s) = 10^{-7} \text{ rad}^2 \cdot \delta(t - s)
\end{align*}
\]
Consider an electric motor with transfer operator

\[ G(p) = \frac{1}{p(p + 1)} \]

from input voltage to actual angular displacement. The motor operates in two disturbance modes:

(i) \[ y(t) = G(p)(u(t) + w(t)) \]

(ii) \[ y(t) = G(p)u(t) + w(t) \]

In both cases we have \( w(t) = \frac{1}{p}v(t) \) where \( v(t) \) is a unit disturbance, for example an impulse.

(a) Realize both cases on state-space form. For case (ii) it is assumed that the states caused by the disturbance are separate from the ones describing the motor dynamics.

(b) For both cases, give examples of physical phenomena that can be modeled with the disturbance \( w(t) \).

(c) Study the two state-space realizations. Are all states controllable? Can states corresponding to \( w(t) \) be made unobservable? Can the influence of \( w(t) \) on \( y(t) \) be eliminated?

Consider the movement of a swing due to the wind. The swing is described by the transfer operator

\[ y(t) = \frac{1}{p^2 + p + 1}u(t) \]
where the output signal $y(t)$ is the angular displacement and the input signal $u(t)$ is the torque about the point of suspension. The influence of the wind can be modeled as

$$u(t) = K v(t)$$

where $v(t)$ is a Gaussian distributed disturbance with the spectrum

$$\Phi_v(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}, \quad \alpha > 0.$$  

$K$ quantifies the strength of the wind and $\alpha$ quantifies the gustiness of the wind.

(a) Does $\alpha$ increase or decrease when the gustiness increases, i.e. when the wind changes direction more frequently?

(b) Derive and interpret conditions on $\alpha$ and $K$ such that the swing has an angular displacement of more than 1.15 at least a quarter of the time. This is equivalent to the output having a variance greater than 1.

Hint:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|b_2(i\omega)^2 + b_1i\omega + b_0|^2}{|i\omega|^3 + a_2(i\omega)^2 + a_1i\omega + a_0|^2} d\omega = \frac{b_2^2a_0a_1 + (b_1^2 - 2b_0b_2)a_0 + b_0^2a_2}{2a_0(-a_0 + a_1a_2)}$$
6 The Closed-Loop System

6.1

For a given system $G$ and a given controller $F$ we have defined four transfer functions as

$$G_{wu} = (I + FG)^{-1}, \quad G_{wu} = -(I + FG)^{-1}F$$

$$G_{w,y} = (I + GF)^{-1}G, \quad G_{w,y} = (I + GF)^{-1}$$

All four transfer functions have to be stable for the closed-loop system to be internally stable.

Show that

$$\begin{pmatrix} G_{wu} & G_{wu} \\ G_{w,y} & G_{w,y} \end{pmatrix} = \begin{pmatrix} I & F \\ -G & I \end{pmatrix}^{-1}$$

6.2

The system

$$G(s) = \frac{s - 1}{s + 1}$$

and the controller

$$F(s) = \frac{s + 2}{s - 1}$$

are used in the feedback connection depicted below.

Compute $G_c$, $T$ and $S$. Are they stable? Is the closed-loop system internally stable?
7 Limitations in Control Design

7.1

Given the system

\[ G(s) = \frac{s - 3}{s + 1}. \]

we want the complementary sensitivity function to be

\[ T(s) = \frac{5}{s + 5}. \]

(a) Compute a controller \( F_r = F_y = F \) which results in this \( T \). Will this controller really work?

(b) Suggest an alternative \( T \), still having the bandwidth 5 rad/s, but resulting in an internally stable system with \( F_r = F_y = F \).

(c) A rule of thumb for control of non-minimum phase systems states that the bandwidth of the closed-loop system cannot realistically be greater than half the value of the non-minimum phase zero. In this case 1.5 rad/s. Have we circumvented this rule of thumb in the above design or does the closed-loop system have any disadvantages?

7.2

A continuous-time system has a zero at \( s = 3 \) and a time-delay of 1.0 second. What is the upper limit of the realistic bandwidth/crossover frequency if the magnitude curve of the open-loop system decreases monotonically?

7.3

Give an example of a system for which there exists no controller having all three properties: a stable closed-loop system, small magnitude of the sensitivity function at low frequencies and small amplification of measurement errors at high frequencies.
7.4

A multivariable system is supposed to attenuate system disturbances \( w \) at least a factor 10 for frequencies under 0.1 rad/s. Furthermore, measurement disturbances \( n \) should be attenuated at least a factor 10 for frequencies above 2 rad/s. Constant system disturbances should be attenuated at least a factor 100 in steady state.

(a) Formulate conditions on the singular values of \( S \) and \( T \) which will guarantee that the requirements are fulfilled.

(b) Translate the specifications into requirements on the loop gain \( GF_y \).

(c) Formulate the requirements using \( \| \cdot \|_\infty \) and frequency weights \( W_S \) och \( W_T \).

(d) Which crossover frequency and phase margin would we expect, having the weights i (b), had the system been a SISO system? What lower bound on \( \| T \|_\infty \) does this result in?

(e) Is this lower bound on \( \| T \|_\infty \) consistent with the requirements in (c)?

7.5

A control system has the sensitivity function \( S \), depicted below

\[
\log |S(i\omega)|
\]

What can be stated about the open-loop system if the surface \( A_2 \) is larger than the surface \( A_1 \)?
7.6

For a certain feedback system we demand that:

(i) output disturbances, with frequencies under 2 rad/s, should be attenuated at least a factor 1000.

(ii) the system should remain stable despite a model uncertainty

\[ |\Delta G| \leq 100|G| \]

for frequencies above 20 rad/s. \( G \) is the frequency response of the nominal system and \( \Delta G \) is the absolute error in the frequency response.

Can this be accomplished using a linear, time-invariant controller?

7.7

We have the following specifications on a SISO system

\[
\begin{align*}
|S(i\omega)| &\leq 10^{-3}, \quad \omega \leq 1 \\
|T(i\omega)| &\leq 10^{-3}, \quad \omega \geq 100
\end{align*}
\]

(a) State two non-constant frequency weights \( W_S \) and \( W_T \) which would guarantee that the specifications are met.

(b) Trying to find a controller fulfilling the design criteria, for example using the methods presented in Chapter 10 in the textbook, we fail. Should this have been anticipated from the very beginning?
8 Controller Structure and Control Design

8.1

Let

\[ G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix} \, . \]

(a) Compute RGA\(G(0)\).

(b) Which input-output pairing should be avoided?

8.2

Given the multivariable system

\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{0.1s + 1} \begin{pmatrix} 0.6 & -0.4 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \, . \]

Assume that we want the controller to be diagonal and that we use the relative gain array (RGA) to decide what input should control what output. Furthermore, assume that we want a crossover frequency of \(\omega_c = 10\) rad/s. Decide how the signals should be paired.

8.3

Study the multivariable system

\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \, . \]

(a) Decide, using RGA analysis, which input signal should control which output signal.
(b) Assume that we want to use decentralized control, i.e. we want a controller on the form

\[ F = W_1 F^{\text{diag}}(s) W_2, \text{ where } F^{\text{diag}}(s) = \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix}. \]

Furthermore, assume that we do not want the steady-state error in one channel to affect the steady-state error in the other channel. Give the structure of a controller \( F(s) \), expressed in \( F^{\text{diag}}(s) \), that will accomplish this.

8.4

Design a controller, using the IMC method, for a stable first order process

\[ G(s) = \frac{K}{\tau s + 1}, \quad \tau > 0. \]

What type of controller do we get? Compute the sensitivity function and the complementary sensitivity function and sketch the Bode plot of the sensitivity function. What does Bode’s integral theorem state for this case?

8.5

Design a controller, using the IMC method, for the system

\[ G(s) = \frac{6 - 3s}{s^2 + 5s + 6}. \]

What type of controller do we get?

8.6

Consider the DC motor

\[ y = \frac{1}{p(p + 1)} u \]
Compute an IMC based controller for this system. Write the controller on the form \( u = -F_y(p)y \), and sketch the Bode plot for \( F_y(p) \). Approximately what type of controller do we get when we want a high bandwidth for the closed-loop system?

8.7

Given the multivariable system

\[
G(s) = \frac{1}{s/20+1} \begin{pmatrix} \frac{9}{s+1} & 2 \\ \frac{6}{s+1} & 4 \end{pmatrix}.
\]

(a) What are the poles and zeros of \( G(s) \)?

(b) Compute an IMC based controller for the system.

8.8

Consider the system

\[
G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}
\]

(Example 1.1 in the textbook)

Show how an IMC based controller can be computed for this system. Give an explicit expression for the corresponding sensitivity function.

8.9

Consider the multivariable system

\[
Y(s) = G(s)U(s)
\]
where

\[ G(s) = \begin{bmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{\alpha}{s+1} & \frac{1}{s+1}
\end{bmatrix} \]

and \( \alpha > 0 \).

(a) Determine the zero of the multivariable system. How does the zero depend on the value of \( \alpha \)?

(b) Assume that one would like to achieve complete decoupling of the system \( G(s) \) such that

\[ G(s)F(s) = \begin{bmatrix}
\frac{1}{(s+1)^2} & 0 \\
0 & \frac{1}{(s+1)^2}
\end{bmatrix} \]

Are there any cases when this is not a good idea? Motivate!

(c) Assume that one instead chooses to use a static decoupling such that \( G(s)F(s) \) is decoupled for \( \omega = 0 \). Are there any values of \( \alpha \) for which this is not a good idea? Motivate!

8.10

Consider the multivariable system

\[ Y(s) = G(s)U(s) \]

where

\[ G(s) = \begin{bmatrix}
\frac{1}{s+2} & \frac{2}{s+4} \\
\frac{1}{s+1} & \frac{1}{s+2}
\end{bmatrix} \]

(a) Determine the RGA at \( \omega = 0 \).
(b) Assume that the system is going to be controlled by a diagonal regulator

\[ U(s) = F(s)(R(s) - Y(s)) \]

where

\[ F(s) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \]

Use the result from a) to judge how successful this will be. Determine also the poles of the closed loop system for the case \( K = 5 \).

(c) How can the problem be modified such that a diagonal \( F(s) \) can be used? Verify that the closed loop system is stable for \( K = 5 \) for the modified problem.
9 Minimization of Quadratic Criteria: LQG

9.1

Consider the system

\[ G(s) = \frac{1}{s - 1} \]

represented on state-space form with noise as

\[ \dot{x}(t) = x(t) + u(t) + v_1(t) \]
\[ z(t) = x(t) \]
\[ y(t) = x(t) + v_2(t) \]

The noises \( v_i(t) \) are white with intensities \( R_i \). We use the criterion

\[ V = \int Q_1 x^2(t) + Q_2 u^2(t) \, dt, \]

and want to find the LQG controller.

(a) Show that the controller is a function of \( \alpha = Q_1/Q_2 \) and \( \beta = R_1/R_2 \) only.

(b) Compute the poles of the closed-loop system as a function of \( \alpha \) and \( \beta \).

9.2

Consider the system

\[ z = \frac{1}{p + 1} u + \frac{1}{p + 1} v \]
\[ y = z + e \]

where \( v \) and \( e \) are unit disturbances with spectra

\[ \Phi_v(\omega) \equiv r_1 \quad \text{respektive} \quad \Phi_e(\omega) \equiv 1. \]

We minimize the criterion

\[ V = \int q_1 z^2(t) + u^2(t) \, dt \]
(a) Compute the loop gain of the feedback connection.

(b) How do \( r_1 \) and \( q_1 \) influence the loop gain?

(c) Sketch the magnitude of the frequency response. What happens when \( r_1 \to \infty \) and when \( q_1 \to \infty \) respectively?

9.3

Consider the double integrator

\[ \ddot{z}(t) = u(t). \]

We want to find a controller such that the criterion

\[ \int_0^\infty (z^2(t) + \eta \cdot u^2(t)) \, dt \]

is minimized for some \( \eta > 0 \). We assume that \( z(t) \) and \( \dot{z}(t) \) are both known (and need not to be estimated).

Where are the poles of the optimal closed-loop system located? How is the control signal affected when \( \eta \) is decreased?

9.4

Consider the antenna in Exercise 5.8. We want to control it and a suitable measure on the performance of closed-loop system is given by the criterion

\[ J = E\{\Theta^2(t) + \rho \mu^2(t)\} \]

where \( \rho \) is a constant we can choose. Derive an optimal control signal and discuss how it is to be combined with the Kalman filter.

9.5

Consider control of the DC motor

\[ G(s) = \frac{1}{s(s + 1)} \]
We want to use the motor together with a system that has a resonance peak at approximately 0.5 rad/s. Other than that, we do not know much about the system. Describe how we can compute an LQG controller with good robustness qualities, i.e. small complementary sensitivity gain, at this frequency.

9.6

A system has static gain $G_0$. It is influenced by system disturbances, with all energy concentrated at zero frequency, i.e.

$$\Phi_\nu(\omega) = \delta(\omega)$$

The reference signal is zero, as is the measurement noise. We choose a controller that minimizes

$$E \{ y^2(t) + \alpha u^2(t) \}$$

What is the value of the sensitivity function at zero frequency?

9.7

Consider the system

$$z = \frac{1}{p + 1}u + \frac{1}{p + 1}\nu$$

$$y = z + e$$

where $\nu$ is noise of very low frequency,

$$\nu = \frac{1}{p + \varepsilon}v,$$

$v$ and $e$ are noises with $\Phi_\nu(\omega) = \Phi_e(\omega) \equiv 1$.

(a) Find a controller that minimizes

$$E \{ z^2 + u^2 \}$$

when $\varepsilon \to 0$.

What is the static gain of the sensitivity function?
(b) Use output-LTR (LTR(y)) to compute L. What is the static gain of the sensitivity function?

9.8

Consider a motor driving two rotating masses connected by a flexible shaft:

![Diagram of motor driving two rotating masses](image)

The angular displacements of the masses are $\varphi_1$ and $\varphi_2$ respectively, and $\omega_1$ and $\omega_2$ are the angular velocities. The moments of inertia are 10 for both masses. The spring rate of the shaft is $k$ and the damping factor is 0.1. The input is the voltage applied to the motor. With the states $x_1 = \varphi_1 - \varphi_2$, $x_2 = \omega_1$ and $x_3 = \omega_2$ we get the state-space representation

\[
\dot{x} = \begin{pmatrix}
0 & 1 & -1 \\
-\frac{1}{2}\omega_0^2 & -0.01 & 0.01 \\
\frac{1}{2}\omega_0^2 & 0.01 & -0.01
\end{pmatrix} x + \begin{pmatrix}
0 \\
0 \\
\omega_0
\end{pmatrix} u
\]

\[
z = \begin{pmatrix}
0 & 0 & 1
\end{pmatrix} x
\]

where

\[
\omega_0^2 = \frac{k}{50}
\]

The Bode plot, when $k = 1$, is shown in the figure below. There is a resonance peak at the frequency $\omega_0$. The spring rate is not exactly known, but has a value close to 1. We want to design a controller that yields a stable closed-loop system despite variations in $k$.

How can the above model be extended with a model for the noise to assure robustness for an uncertain value of $k$ when we use LQG controller design? Give an actual example of such an extended system.
9.9

Consider the system

\[ \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t) \]

Show that

\[ u(t) = -\begin{pmatrix} 2 & -3 \end{pmatrix} x(t) \]

cannot be an optimal state feedback for any quadratic criterion on the form

\[ \min \int (x^T(t)Q_1x(t) + Q_2u^2(t)) \, dt \]

where \( Q_1 \) is a positive definite matrix.

9.10

Consider the system

\[ \begin{align*}
\dot{x} &= \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u \\
y &= (1 \ 1) x
\end{align*} \]
We want to minimize the criterion

\[ V(T) = \int_0^T x^T(t)x(t) + u^2(t)dt \]

Is it possible to find a state feedback \( u = -Lx \) such that \( V(T) < \infty \) when \( T \to \infty \)?

9.11

The figure below shows a simple electrical circuit.

Introduce the state variables \( x_1 = V_C \) and \( x_2 = i \). With the component values

\[ R = 5 \, \Omega, \quad L = 0.1 \, H, \quad C = 1000 \, \mu F \]

we get the state-space representation

\[
\begin{align*}
\dot{x}(t) &= \begin{pmatrix} 0 & 1000 \\ -10 & -50 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 10 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)
\end{align*}
\]

Compute a state feedback that minimizes

\[ J = \int_0^\infty (x_2^2(t) + 0.01u^2(t)) \, dt \]

This criterion aims at limiting the power loss without getting too large signals.
9.12

A system has the state-space representation

\[
\dot{x}(t) = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 0.5 \\ 0 & 0 & A \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(t)
\]

\[
z(t) = (1 \ 0 \ 0) x(t)
\]

\[
y(t) = z(t) + e(t)
\]

where \(e(t)\) are unit disturbances.

The controller, a feedback from reconstructed states, minimizes

\[
E \left[ z^2(t) + u^2(t) \right]
\]

How does the value of \(A\) affect the sensitivity function?

9.13

A simplified model for how the elevator angle affects the movements of an airplane is given by

\[
\dot{x} = \begin{bmatrix} -0.01 & 0.03 & -10 \\ 0 & -1 & 300 \\ 0 & 0 & -0.5 \end{bmatrix} x + \begin{bmatrix} 4 \\ -20 \\ -10 \end{bmatrix} u
\]

where

\[
x = \begin{bmatrix} \text{roll angle} \\ \text{yaw angle} \\ \text{pitch-angle velocity} \end{bmatrix}
\]

In particular we are interested in the control of the pitch-angle velocity and choose the controlled variable to be

\[
z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x
\]

All state variables are measured

\[
y = x + e
\]
We want to design a feedback from reconstructed states using LQG methodology. It is especially important that the sensitivity function has a small gain for frequencies around 1 rad/s. Show how to modify the model of the airplane to achieve such a sensitivity function.

9.14

Consider the system

\[ \dot{x}(t) = \alpha x(t) + u(t) \quad x(0) = x_0 \]  

(1)

The system is controlled by the feedback

\[ u(t) = -Lx(t) \]  

(2)

where \( L \) is chosen such that

\[ J = \int_0^\infty x^2(t) + \rho u^2(t) dt \]  

(3)

is minimized.

(a) Determine \( L \) as function of \( \rho \) and \( \alpha \).

(b) If it is desired to keep \( u(t) \) small, this can be achieved by choosing \( \rho \) large. What is the resulting \( L \) when \( \rho \rightarrow \infty \)? Consider, for example, the cases \( \alpha = 1 \) and \( \alpha = -1 \), respectively. Why is it not optimal to choose \( L = 0 \), i.e. \( u(t) = 0 \), in both cases?

9.15

An electrical motor has the transfer functions

\[ Y(s) = \frac{1}{s(s + 1)}U(s) \]

and it is controlled using state feedback

\[ u(t) = -Lx(t) \quad (r(t) = 0) \]
where $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. The gain vector $L$ is determined by
minimizing the criterion

$$J = \int_{0}^{\infty} x^T(t)Q_1x(t) + Q_2u^2(t)dt$$

Figure 1 shows the simulation results when the system starts in the initial condition $x(0) = (1\ 1)^T$ for some different choices of $Q_1$ and $Q_2$. Combine the figures with the choices of matrices.

(i) $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = 0.1$

(ii) $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, $Q_2 = 1$

(iii) $Q_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = 0.1$

(iv) $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = 1$
9.16

Consider the simplified description of an aircraft in the figure below.

Using the state space variables
\( x_1(t) = \alpha(t) \) angle of attack (rad)
\( x_2(t) = \dot{\theta}(t) \) pitch rate (rad/s)
\( x_3(t) = \theta(t) \) pitch angle (rad)
\( x_4(t) = h(t) \) height (deviation from an operating point)

the input signal
\[ u(t) = \delta(t) \] control surface angle (rad)

and output signal
\[ y(t) = h(t) \] height (hundreds of meters)

the system is described by the state space model
\[ \dot{x} = Ax + Bu \quad y = Cx \]

where
\[
A = \begin{bmatrix}
-0.17 & 1 & 0 & 0 \\
-0.56 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2.22 & 0 & 2.22 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0.011 \\
0.56 \\
0 \\
0
\end{bmatrix}, \quad C = (0 \ 0 \ 0 \ 1).
\]

(a) Is the system asymptotically stable?

(b) Assume that the system has the initial state
\[ x_0 = (0 \ 0 \ 0.1 \ 1)^T \]

and that the system is controlled by the state feedback
\[ u = -Lx \]

where the gain vector \( L \) is chosen such that the criterion
\[
\int_0^\infty x^T(t)Q_1x(t) + u^T(t)Q_2u(t) \, dt
\]
is minimized. Assume that the matrices are chosen as
\[
Q_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad Q_2 = 1
\]

Determine the poles of the closed loop system. Simulate the closed loop system.
(c) Assume that $Q_2$ is varied. How does that affect the location of the closed loop poles and the properties of $x$ and $u$?

(d) Assume now that the following conditions shall be fulfilled:

- $|x_1| < 0.2$ all the time.
- $|x_4| < 0.1$ after 25 seconds.
- $|u| < 0.5$ after one second.

Determine $Q_1$ and $Q_2$ such that these conditions are satisfied. What is the resulting location of the closed loop poles?

9.17

The figure below illustrates a system consisting of a ball on a plane. The variable $r$ denotes the position of the ball relative to the center of the plane, and $\alpha$ represents the angle of the plane. The input signal is the torque that rotates the plane.

![Figur 2: Ball on plane.](image)

The system is represented by the state variables

- $x_1(t)$ - position, $r(t)$
- $x_2(t)$ - velocity, $\dot{r}(t)$
- $x_3(t)$ - plane angle, $\alpha(t)$
- $x_4(t)$ - plane angular velocity, $\dot{\alpha}(t)$

and torque is the input signal $u(t)$. The state space model is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
where

\[
A = \begin{pmatrix} 
0 & 1 & 0 & 0 \\
0 & 0 & -7 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad
B = \begin{pmatrix} 
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}, \quad
C = \begin{pmatrix} 
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(a) Assume that the system starts in the initial state

\[x(0) = (0.1 \ 0 \ -0.1 \ 0)^T\]

i.e. the ball is positioned to the right of the center, and the plane leans downwards on the right side. Assume that all state variables can be measured. Determine a state feedback such that the following requirements are fulfilled:

- \(|x(t)| \to 0\) when \(t \to \infty\).
- \(|x_1(t)| \leq 0.2\) \(\forall t\).
- \(|u(t)| \leq 2.5\) \(\forall t\).

Determine also the absolute value of the poles of the closed loop system.

(b) Verify that all sensors that measure the states have to work in order to obtain a stable closed loop system.

**Hint:** The characteristic equation of the closed loop system is given by

\[\lambda^4 + l_4\lambda^3 + l_3\lambda^2 - 7l_2\lambda - 7l_1 = 0\]

Missing a sensor is equivalent to setting the corresponding feedback \(l_i\) to zero.
10 Loop Shaping

10.1

Consider the system
\[ y = \frac{1}{p + 1} u \]
We want to create a closed-loop system with \( S, T \) and \( G_{wu} \), such that
\[
\int \left| \frac{S(i\omega)}{i\omega} \right|^2 + |0.5 T(i\omega)|^2 + |5 G_{wu}(i\omega)|^2 d\omega
\]
is minimized. Compute the controller.

10.2

Consider the system
\[ y = \frac{1}{p + 1} u \]
We want to create a closed-loop system with \( S, T \) and \( G_{wu} \), such that
\[
|S(i\omega)| < \gamma \omega \\
|T(i\omega)| < 2\gamma \\
|G_{wu}(i\omega)| < 0.2\gamma
\]
Write down the equations that determines the controller.

10.3

Consider the SISO system \( G(s) \) with state-space realization
\[
\dot{x} = Ax + Bu \\
y = Cx
\]
We want to use loop shaping with the weights
\[ W_S = \frac{1}{s}, \quad W_T = 1, \quad W_u = 1 \]
(a) State the equations that determine the optimal controller in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ respectively.

(b) Explicitly write down the observer for the extended state vector and show that the optimal controller can be written as

$$u(t) = \frac{\alpha}{1 + L(pI - A)^{-1}B} \int_0^t y(\tau) d\tau$$

for some $L$, where $\alpha = 1$ for the $\mathcal{H}_2$ controller and $\alpha > 1$ for the $\mathcal{H}_\infty$ controller. State the equation determining $L$.

(c) Show that the controller will have a pole at the origin unless the system does itself have a pole at the origin.

10.4

Once again consider the system in Exercise 9.8.

(a) Suggest frequency weights $W_S$, $W_T$ and $W_u$, for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design, such that we get robustness against uncertain values of $k$.

(b) State the extended system from $u$ and $w$ to $z$ on state-space form.

10.5

A DC-motor has transfer function

$$G(s) = \frac{1}{s(s + 1)}$$

and it going to be controlled using proportional feedback

$$U(s) = K(R(s) - Y(s))$$

The properties of the closed loop system is specified via the requirement

$$|S(i\omega)| < |W_S^{-1}(i\omega)| \quad \forall \omega$$

The figure below shows three alternatives for the weight function $W_S^{-1}(i\omega)$. Which alternative is the best? Motivate the answer.
Figur 3: Suggestions for $|W_S^{-1}(i\omega)|$.

10.6

The system

$$Y(s) = \frac{1}{s + 1} U(s)$$

is going to be controlled by the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

(a) Derive $S(s)$, $T(s)$ and $G_{ru}(s)$ respectively, i.e. the sensitivity function, the complementary sensitivity function and the transfer function from reference to input signal.

(b) The properties of the control system are specified using the weight function according to

$$|S(i\omega)W_S(i\omega)| < 1 \quad \forall \omega$$

$$|T(i\omega)W_T(i\omega)| < 1 \quad \forall \omega$$

$$|G_{ru}(i\omega)W_u(i\omega)| < 1 \quad \forall \omega$$
The figures below show three suggestions for weight functions $W_S$, $W_T$, and $W_U$. Two of the alternatives are unrealistically or incorrectly specified. Which are the two incorrect alternatives? Motivate the answer.

(c) Consider the alternative in b) that is realistically specified. Is it possible to choose $K$ such that all requirements are fulfilled?

Figur 4: Alternative I
Figur 5: Alternative II

Figur 6: Alternative III
12 Stability of Nonlinear Systems

12.1

Given the nonlinear differential equation

\[ \ddot{y} + 0.2(1 + \dot{y}^2)\dot{y} + y = 0 \]

let the state variables be \( x_1 = y \) and \( x_2 = \dot{y} \). Try to show that the origin is a stable equilibrium by using the Lyapunov function candidate

\[ V = \frac{1}{2}(x_1^2 + x_2^2). \]

12.2

Consider the system

\[ \begin{align*}
\dot{x}_1 &= \sin x_1 + x_2^3 \\
\dot{x}_2 &= x_1 - x_2
\end{align*} \]

Is it possible to use the function

\[ V(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 \]

to prove Lyapunov stability for the above system? Motivate your answer.

12.3

A nonlinear function lies in the sector
According to the circle criterion, what circle in the complex plane corresponds to this nonlinearity?

12.4

A nonlinear system is described by the following block diagram

$$\Sigma r \quad G(s) \quad f$$

where $G(s)$ is a linear system and the static nonlinearity $f$ is given in the figure below (the saturations at $-1$ and $1$ extends to $-\infty$ and $\infty$).
What assumptions on $G(s)$ must be fulfilled in order to prove that the feedback system is stable according to the circle criterion?

12.5

Consider the system below.

The nonlinearity $f$ is such that $u_2$ has the same sign as $u_1$ but is otherwise not known. For what values of $K > 0$ is the feedback system stable according to the circle criterion?
Consider the swing depicted below.

The movement of the swing is described by the equation

\[ J \frac{d^2 \Phi}{dt^2} + mg \ell \sin \Phi = 0 \]

where \( m \) is the mass and \( J \) is the moment of inertia. The swing can be controlled by alternating between bending and stretching the knees while standing on the swing. The control signal is the location of the center of gravity \( \ell \). We assume that \( J \) is constant.

Show that the control signal

\[ \ell = \ell_0 + \varepsilon \Phi \dot{\Phi}, \quad \varepsilon > 0 \]

will bring the swing to rest in \( \Phi = 0 \).

12.7

The block diagram below is given.
We have that

\[ H(s) = s \quad \text{and} \quad G(s) = \frac{1}{(s+1)(s+2)}. \]

How shall the feedback coefficients \(a\) and \(b\) be chosen to guarantee Lyapunov stability?

**Hint:** Use a quadratic Lyapunov function candidate.

---

12.8

A servo system contains a nonlinearity where the relationship between the input signal \(u\) and the output signal \(y\) is

\[ y = u + \arctan(u) \]

What requirements on the linear part of the servo system must be fulfilled in order to prove stability using the circle criterion?
A simplified model for the movements of an airplane is given by

\[
\dot{x} = \begin{bmatrix}
-0.01 & 0.03 & -10 \\
0 & -1 & 300 \\
0 & 0 & -0.5 \\
\end{bmatrix} x + \begin{bmatrix}
4 \\
-20 \\
-10 \\
\end{bmatrix} u
\]

All states are measured and the control signal is

\[
\tilde{u} = -Lx + \tilde{r}
\]

where \(L\) is the feedback that minimizes

\[
\int (x^T(t)Q_1x(t) + u^2(t)) \, dt
\]

for \(Q_1 = 10 \cdot I\)

The requested control signal \(\tilde{u}\) is different from the actual \(u\), due to the hydraulic servo dynamics. The relationship between \(\tilde{u}\) and \(u\) is

![Graph showing the relationship between \(u\) and \(\tilde{u}\)]

Will the closed-loop system be stable? Motivate your answer.
13 Phase Plane Analysis

13.1

Given the differential equation

\[ \ddot{y} - (0.1 - \frac{10}{3} \dot{y}^2)\dot{y} + y + y^2 = 0 \]

Find and classify the singular points.

13.2

Draw the phase portrait of the depicted position servo.

The position is measured using an E-transformer, which can be described as a dead zone. Assume that \( K > \frac{B^2}{4} \).

13.3

The following system is given
(a) With zero input signal the output of the relay is +1 or −1, depending on the history of the input signal. The relay does not switch until the input signal has changed polarity.

Draw a phase portrait of the system.

(b) Due to imperfections the actual feedback loop is

Draw the phase portrait of this system.

13.4

Linus is on his way home after an exam. On the highway outside of Linköping a gust of wind makes the car drift from the desired path. Your task is to, using phase plane analysis, decide how the movement of the car will progress. Will it return to the desired path? If the car has a constant speed in the direction of travel the system can be described by the following block diagram
The torque applied to the steering wheel is $u$. The backlash comes from a gear unit in the steering. The output signal $y$ is the deviation from the desired path. $G(s)$ is the transfer function from Linus’ visual perception to the torque he applies to the steering wheel.

Distinguish between the cases:

(a) $G(s) = 1$ (there was a party after the exam)

(b) $G(s) = 1 + s$ (there was not a party after the exam)

13.5

A simple ecological system consists of two species of fish. The first kind eats algae and the second kind eats the first kind. Let $x_1$ denote the number of algae eating fish and $x_2$ denote the number of predatory fish. Then we have

$$
\dot{x}_1 = 2x_1 - \frac{x_1 x_2}{1 + \frac{1}{6}x_1} - 0.2x_1^2
$$

$$
\dot{x}_2 = -3x_2 + \frac{x_1 x_2}{1 + \frac{1}{6}x_1}
$$

(a) From these equations, calculate the stationary points.
(b) Classify the stationary points and sketch the phase portraits in a surrounding of them. It is sufficient to consider a linearised version of the equations.

(c) Without any further calculations, merge the phase portraits you have made around the stationary points in a fashion that seems reasonable. Only consider \( x_1 > x_2 > 0 \).

An interpretation of the given equations is:

If the algae eating fish have an infinite amount of food and lack enemies, their number will grow exponentially as

\[
\dot{x}_1 = 2x_1
\]

As there is a limited amount of algae the growth saturates according to

\[
\dot{x}_1 = 2x_1 - 0.2x_1^2.
\]

If there are predatory fish \( x_2 \) present the algae eaters will be devoured at the rate

\[
\frac{x_1 x_2}{1 + \frac{1}{6}x_1}
\]

The interpretation of this term is that if \( x_1 \) is large every predatory fish can eat until it is full. This corresponds to 6 algae eating fish per time unit. On the other hand, if the number \( x_1 \) is relatively small the predatory fish will eat less.

The second equation says that if the supply of food is unlimited \( (x_1 = \infty) \) the predatory fish will multiply according to

\[
\dot{x}_2 = 3x_2.
\]

If food is lacking \( (x_1 = 0) \) the predatory fish will expire as

\[
\dot{x}_2 = -3x_2
\]

13.6

A mass is suspended from a spring. Its position \( y(t) \) satisfies the differential equation

\[
\ddot{y}(t) + y(t) = f(t)
\]
where \( f(t) \) is an external force acting on the mass. Draw a phase portrait of the system when

\[
f(t) = \begin{cases} 
-1 & \text{if } \dot{y}(t) > 0 \\
+1 & \text{if } \dot{y}(t) < 0 
\end{cases}
\]

Will the system reach an equilibrium?

13.7

Consider the system

\[
\dot{x} = \begin{pmatrix} 
-x_1^3 + u \\
x_1 
\end{pmatrix}
\]

(a) Sketch a phase portrait when \( u = 0 \).

(b) Use the Lyapunov function

\[
V(x) = x_1^2 + x_2^2
\]

to compute a control signal

\[
u = f(x_1, x_2)
\]

which will make the origin globally asymptotically stable. Sketch a phase portrait, in a neighborhood of the origin, for the closed-loop system.
14 Oscillations and Describing Functions

14.1

Consider the feedback control system including an input saturation according to the figure below.

(a) Investigate the stability of the system. If a periodical solution exists, determine its frequency and amplitude.

(b) Build a simulation model of the control system and investigate the validity of the results from a).

14.2

A temperature control system, depicted below, contains a relay with dead zone.

\[ r(t) = 0 \]
\[ \Sigma \]
\[ u(t) \]
\[ G_0(s) \]
\[ y(t) \]
\( G_0(s) = \frac{1}{s(1+s)^2}, \) \( \pm D \) is the width of the dead zone and \( \pm H \) is the output level of the relay. The values of the dead zone and output level are such that a stable oscillation just barely can exist. If \( H \) is increased or if \( D \) is decreased an oscillation will not be possible. The amplitude of the oscillation is 2.5 units. Compute \( D, H \) and the frequency of the oscillation. The describing function for a relay with dead zone is

\[
\text{Re}\{Y_N(C)\} = \frac{4H}{\pi C} \sqrt{1 - D^2/C^2}, \quad C \geq D
\]

\[
\text{Im}\{Y_N(C)\} \equiv 0
\]

### 14.3

A relay servo is given by

\[
\theta_{\text{ref}} \rightarrow \Sigma \rightarrow u \rightarrow \frac{K}{s(s + 1)^2} \rightarrow \theta
\]

\( -L(s) \)

The gain \( K \) is strictly positive.

(a) The feedback used is \( L(s) = 1 \). Show that there is an oscillation for all values of \( K \).

(b) To avoid too much wear on the system we do not want the amplitude of the oscillation in \( \theta \) to be greater than 0.1. For what values of \( K \) is this fulfilled?

(c) We want to use a gain \( K \) that is larger than what is possible in (b). State a feedback \( L(s) \) with \( L(0) = 1 \) that makes this feasible. No details are necessary. Just motivate why the feedback should solve the problem.
14.4

Consider the nonlinear system

\[ u \rightarrow \Sigma \rightarrow -H(s) \rightarrow \frac{1}{s(s+1)(s+2)} \rightarrow y \]

(a) If proportional control is used, i.e. \( H(s) = 1 \), a stable oscillation occurs. Find the amplitude and frequency of the oscillation.

(b) To eliminate the oscillation we use proportional and derivative control, i.e. \( H(s) = 1 + Ks \). Show how \( K \) can be chosen to eliminate the oscillation.

14.5

Consider the feedback control system where a motor is controlled using a relay with hysteresis.

\[ \theta_r = 0 \rightarrow \Sigma \rightarrow u(t) \rightarrow \frac{1}{s(s+1)} \rightarrow \theta \]

(a) Investigate the stability of the system using the describing function method. If a periodical solution exists, determine its frequency and amplitude.
(b) Build a simulation model of the control system and investigate the validity of the results from a).

(c) Introduce suitable state variables and sketch a phase portrait.

14.6

Consider the following servo system

\[
\begin{align*}
    r &= 0 \\
    \Sigma &
        \quad e \quad K(1 + \frac{1}{Ts} + T_Ds) \quad \tilde{u} \quad u \quad \frac{1}{s^2} \quad y
\end{align*}
\]

The PID controller has \( K = 2 \), \( T_I = 2 \) and \( T_D = 0.5 \).

(a) The tuning of the controller was done assuming that the amplifier has the transfer function 1. Show that, if this assumption is true, this results in an asymptotically stable closed-loop system.

(b) The actual amplifier contains a saturation

\[
\begin{align*}
    u &
        \quad 1 \\
    \tilde{u} &
        \quad -1 \\
    -1 &
        \quad 1
\end{align*}
\]