Nonlinear control

Lecture 5. Lyapunov based design

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Review of course

- Geometric control theory
  - input-output linearization
  - controller canonical form
  - observer canonical form

- Lyapunov theory
  - Stability results
  - Passivity
  - Circle and Popov criteria

Lyapunov design

1. Control Lyapunov functions
2. Back-stepping
3. Forwarding
4. Observers
5. (Disturbance supression)
6. (Passivity based control)

Control Lyapunov functions

$V$ is a **Control Lyapunov function** if

for every $x$ there is some $u$ so that $V_x f(x,u) < 0$

Choose $u = k(x)$ so that

$V_x f(x,k(x))$ negative definite

Might be difficult to find nice $k$. 
Control Lyapunov function. Example

The spring $k$ has cubic stiffness.

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + u
\end{align*}
\]

Possible control Lyapunov functions:

\[
V_1 = \frac{x_1^4}{4} + \frac{x_2^2}{2}, \quad V_2 = \frac{x_1^2}{2} + \frac{x_2^4}{4} + \frac{x_3^2}{2}
\]

Back-stepping with Lyapunov functions

If the actuator is first order one can take $u_{\text{act}} = z$:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)z \\
\dot{z} &= a(x, z) + b(x, z)u
\end{align*}
\]

Suppose we find a control law $z = k(x)$ and a Lyapunov function $V(x)$ so that

\[
V_x(x)(f(x) + g(x)k(x)) = -W(x), \quad V, W \text{ positive definite}
\]

This control law can then be extended (the "step back") to a control law for $u$, using a control Lyapunov function, e.g.

\[
V_c(x, z) = V(x) + \frac{1}{2}(z - k(x))^2
\]

A typical block structure

- Design a controller assuming $u_{\text{act}}$ is the control signal
- Step back to the real control signal $u$ and extend the controller design ("backstepping").

Repeated backstepping

Systems in feedback form:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n) + g_n(x_1, \ldots, x_n)u \\
\end{align*}
\]

Repeated backstepping is easily done for the structure to the left. It can also be generalized to the structure to the right.
Back-stepping

Basic advantage: Not necessary to cancel terms that make $\dot{V}_e$ negative. Many opportunities for creative extensions.

Tutorial, theory and applications in:

Forwarding

Extending the Lyapunov function when an integrator is added to the output.

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= a(x) + b(x)u \\
\end{align*}
\]

Suppose positive definite functions $V$, $W$ and a control law $k$ are known so that

\[
V_e(x)(f(x) + g(x)k(x)) = -W(x)
\]

Let $\zeta = \phi(x,z)$ be constant when $u = k(x)$. Then

\[
V_e(x,z) = V(x) + \frac{1}{2} \zeta^2
\]

is a suitable control Lyapunov function.

Forwarding example

\[
\begin{align*}
\dot{z} &= x^2 + u \\
\dot{x} &= -x + u \\
\end{align*}
\]

Stabilization of $x$-system: $u = k(x) = 0$.
Coordinate change: $\zeta = z + \frac{x^2}{2}$
Resulting control law:

\[
u = -x - (1 + x)\zeta
\]

Result, forwarding example

Forwarding control for: $x(0) = 6.5, z(0) = 1$

Left diagram: $x$, right diagram: $z$
full forwarding controller: red
linear part of the controller: blue
The high gain observer

\[ \dot{x} = Ax + B\phi(x) + g(x)u, \quad y = Cx \]

\[ A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad C = [1 \ 0 \ \ldots \ 0] \]

The observer is

\[ \dot{\hat{x}} = A\hat{x} + B\phi(\hat{x}) + g(\hat{x})u + K(y - C\hat{x}) \]

A Lyapunov function for the observer

The observer error is

\[ \dot{\hat{x}} = (A - KC)\hat{x} + B(\phi(x) - \phi(\hat{x})) + (g(x) - g(\hat{x}))u \]

With \( K = S^{-1}C^{T} \) and \( S \) given by

\[ A^{T}S + SA - C^{T}C = -\theta S \]

the function

\[ V = \hat{x}^{T}S\hat{x} \]

is a Lyapunov function, if \( \theta \) is large enough.

State feedback via observer

State feedback: \( \dot{x} = f(x) + g(x)u, \ u = k(x) \)
Lyapunov function \( \dot{V} \):

\[ \dot{V} = V_{x}(x)(f(x) + g(x)k(x)) \leq -q(x) \leq 0 \]

Observer: observer error \( \tilde{x} = x - \hat{x} \)
Lyapunov function \( V_{e} = \|\tilde{x}\|_{Q}^{2} \) for some norm \( \|Q\) :

\[ \dot{V}_{e} = -q_{e}(x) \leq 0 \]

Is \( W(x, \tilde{x}) = V(x) + V_{e}(\tilde{x}) \) a Lyapunov function for the closed loop system?

Disturbances

\[ \dot{x}_{1} = x_{2} \]
\[ x_{2} = -x_{1} - x_{3}^{2} - x_{2} + u + w \]
Disturbance suppression

Result of disturbance compensation

To the left: $V(t)$, to the right: $x_1(t)$.
Uncontrolled system without disturbance: dotted, uncontrolled system with disturbance: dashed, controlled system with disturbance: solid.

Disturbance example

Disturbance $w$, control signal $u$, and filtered control signal

Passivity

Passive system:

$$\int_0^T u^T y \, dt + \gamma(x(0)) \geq 0$$

Design a feedback law

$$u = v - k(x)$$

so that the system is still passive from $v$ to $y$.

Passivity and feedback

If two passive systems are connected in a feedback loop, the resulting system is passive:

$$u \rightarrow S_1 \leftarrow u_1 \rightarrow S_2 \leftarrow y_2 \rightarrow y$$