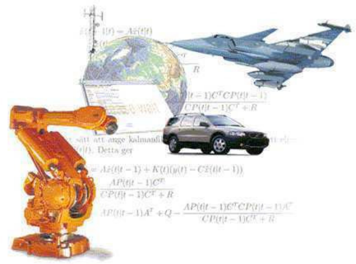
Dynamic Systems

Lecture 6. Polynomial Matrix Descriptions



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Matrix Fraction Descriptions (MFDs)

For a system (not necessarily in state space form)

$$E\dot{x} = Ax + Bu, \quad y = Cx$$

with n variables, m inputs, and p outputs, the transfer function

$$G(s) = C(sE - A)^{-1}B$$

is a $p \times m$ matrix whose elements are rational functions.
A closer analogy to the SISO case is the *matrix fraction* description

$$G(s) = N_R(s)D_R^{-1}(s) \text{ or } G(s) = D_L^{-1}(s)N_L(s)$$

where N_R, D_R, N_L, D_L are *polynomial matrices*.

Left and right MFDs, example

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \end{pmatrix}$$

Left MFD $\quad G(s) = \underbrace{\big((s+1)(s+3)\big)}_{D_L}^{-1} \underbrace{\begin{pmatrix} s+3 & 2(s+1) \end{pmatrix}}_{N_L}$

Right MFD $\quad G(s) = \underbrace{\begin{pmatrix} 1 & 2 \end{pmatrix}}_{N_R} \underbrace{\begin{pmatrix} s+1 & 0 \\ 0 & s+3 \end{pmatrix}}_{D_R}^{-1}$

Note that the dimensions of D_L and D_R are not the same. Also note that

$$\det D_L(s) = \det D_R(s) = (s+1)(s+3)$$

State space and descriptor systems

A state space description where all state variables are regarded as outputs

$$\dot{x} = Ax + Bu, \quad y = x$$

is directly represented as a left MFD:

$$G(s) = (sI - A)^{-1}B$$

This is true also for a descriptor or DAE representation

$$E\dot{x} = Ax + Bu, \quad y = x$$

with

$$G(s) = (sE - A)^{-1}B$$

Common factors

$$N(s) = \tilde{N}(s)R(s), \quad D(s) = \tilde{D}(s)R(s)$$

- The polynomial matrix R is said to be a *common right divisor*.
- $N(s)D^{-1}(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$
- If R can be written as $R = \tilde{R}S$ for every common right divisor S, then R is a *greatest common right divisor* (gcrd).
- A polynomial matrix whose inverse is also polynomial is a trivial common factor. Such a matrix is called *unimodular*.
- If a gcrd of N and D is unimodular then N and R are said to be *right coprime*.
- Common left divisor, gcld, and left coprime are defined analogously for left MFDs.

Unimodular matrices

Fact $P(s)$ unimodular $\Leftrightarrow \det P(s) =$ const. $(\neq 0)$
Examples of unimodular matrices.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & a(s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When multiplying a matrix from the left they correspond to
○ an exchange of the first two rows
○ an addition of $a(s) \times$ (second row) to the first row
○ multiplication of first row by 5
Elementary row operations thus correspond to multiplication by unimodular matrices from the left.

Multiplication from the right: corresponding column operations

Hermite form

For a polynomial matrix $P(s)$ with independent columns it is possible to find a unimodular matrix $U(s)$ so that

$$U(s)P(s) = \begin{pmatrix} \times & \times & \dots & \times \\ 0 & \times & \dots & \times \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \times \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where the diagonal elements are nonzero monic polynomials of higher degree than the elements above. This is called **Hermite form**.

More on Hermite form

Both Mathematica and Maple have packages for polynomiaol matrices that compute the Hermite form.

For a matrix with independent **rows**, an analogous triangular form can be obtained by multiplying from the **right** with a unimodular matrix.

Finding a gcrd

A gcrd for N_R, D_R in $G(s) = N_R(s)D_R^{-1}(s)$:
Use e.g. Hermite transformation to get

$$\begin{pmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{pmatrix} \begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix} = \begin{pmatrix} R(s) \\ 0 \end{pmatrix}$$

with the U-matrix unimodular.
With $V = U^{-1}$:

$$\begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix} = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix} \begin{pmatrix} R(s) \\ 0 \end{pmatrix}$$

1. R is a gcrd of N_R and D_R.
2. V_{11} nonsing., $\det V_{11} = \text{const.} \cdot \det U_{22}$.
3. $G(s) = V_{21}(s)V_{11}(s)^{-1}$ coprime
4. $G(s) = -U_{22}(s)^{-1}U_{21}(s)$ coprime
.

A technical result

Lemma Let $P(s)$ be a $p \times r$ pol. matrix and $Q(s)$ a nonsingular $r \times r$ pol. matrix. The following are equivalent.

1. P and Q are right coprime.
2. There exist pol. matrices $X(s)$ ($r \times p$) and $Y(s)$ ($r \times r$) such that the following **Bezout identity** is satisfied:

$$X(s)P(s) + Y(s)Q(s) = I$$

3. For every complex s:

$$\text{rank} \begin{pmatrix} Q(s) \\ P(s) \end{pmatrix} = r$$

An analogous lemma holds for left coprime matrices.

Example

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} s+2 & 1 \\ s & 2s+1 \end{bmatrix} \begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s^2+3s+2 \end{bmatrix}^{-1}$$

Hermite transformation gives

$$\begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s^2+3s+2 \\ s+2 & 1 \\ s & 2s+1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & s+1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U^{-1} = V = \begin{bmatrix} s^2+3s+2 & -s-2 & 0 & s+1 \\ 0 & s+2 & 1 & 0 \\ s+2 & -1 & 0 & 1 \\ s & 1 & 0 & 1 \end{bmatrix}$$

Example cont'd.

A right coprime MFD:

$$\underbrace{\begin{pmatrix} s+2 & -1 \\ s & 1 \end{pmatrix}}_{N_R} \underbrace{\begin{pmatrix} (s+1)(s+2) & -s-2 \\ 0 & s+2 \end{pmatrix}^{-1}}_{D_R}$$

Using U_{21} and U_{22} gives the left coprime MFD:

$$\underbrace{\begin{pmatrix} (s+1)(s+2) & 0 \\ 2s^2+5s+2 & -s-2 \end{pmatrix}^{-1}}_{D_L} \underbrace{\begin{pmatrix} s+2 & 1 \\ 2s+2 & 0 \end{pmatrix}}_{N_L}$$

Note that

$$\det D_R(s) = -\det D_L(s) = (s+1)(s+2)^2$$
$$\det N_R(s) = -\det N_L(s) = 2s+2$$

Almost uniqueness of coprime MFDs

Theorem If

$$G(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$$

with both MFDs being right coprime,
then there is a unimodular matrix U such that

$$N_1(s) = N_2(s)U(s), \quad D_1(s) = D_2(s)U(s)$$

An analogous result holds for left coprime MFDs.

Comparing left and right MFDs

Theorem If

$$G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s)$$

with both MFDs being coprime,
then there is a constant $k \neq 0$ such that

$$\det D_R(s) = k \det D_L(s)$$

Strictly proper systems

We have shown:
Theorem $G(s)$ has time-invariant finite-dimensional realization $\Leftrightarrow$ each element is a strictly proper rational function

Is

$$\begin{bmatrix} 3s+2 & 1 \\ 2s+3 & 1 \end{bmatrix} \begin{bmatrix} s^2+2s & s+1 \\ s^2+s & s+1 \end{bmatrix}^{-1}$$

strictly proper?

Row and column degrees

$$G(s) = N_R(s)D_R(s)^{-1}$$

Definitions:

- k_i: highest degree, i:th column of $D_R(s)$.
- $G(s)$ proper: $\lim_{s\to\infty} G(s)$ finite
- $G(s)$ strictly proper: $\lim_{s\to\infty} G(s) = 0$

Easy results:

- G strictly proper $\Rightarrow$ each $k_i >$ degree of corresponding column in N_R
- G proper $\Rightarrow$ each $k_i \geq$ degree of corresponding column in N_R
- deg $\det D_R(s) \leq \sum k_i$

Column and row reduced systems

- D_{hc}: matrix of columnwise highest degree coefficients.
- D_R is said to be *column reduced* if deg $\det D_R(s) = \sum k_i \Leftrightarrow D_{hc}$ is nonsingular.

Theorem Let D_R be column reduced. Then $G = N_R D_R^{-1}$ is strictly proper (proper) if and only if, for each i, column i of N_R has a maximum degree $< k_i$ ($\leq k_i$).

An analogous statement holds for left MFDs (row reduced)

Getting column reduced descriptions

It is possible to perform column operations, or equivalently to multiply from the right by a unimodular U, so that in the new description

$$\tilde{N}_R = N_R U, \quad \tilde{D}_R = D_R U$$

$\tilde{D}_R$ is column reduced.