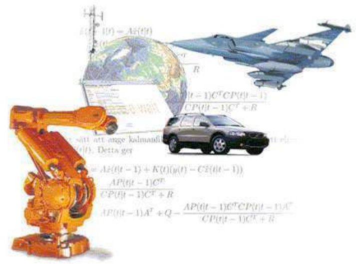
Linear Systems

Lecture 4. Input-output relations



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Linear input-output relations

State space description:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t)$$

Input-output relation ($x(t_0) = 0$):

$$y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

Impulse response:

$$h(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$$

A, B, C constant:

$$h(t,\tau) = Ce^{A(t-\tau)}B = h(t-\tau,0), \quad G(s) = C(sI-A)^{-1}B$$

Discrete time Volterra series

Discrete time nonlinear causal system:

$$y(t) = F(t, u(0), u(1), \ldots, u(t)), \quad t = 0, 1, 2, \ldots$$

F smooth – Taylor expansion

$$y(t) = y_0(t) + \sum_{j=0}^{t} g_1(t,j)u(j) + \sum_{j=0}^{t}\sum_{k=0}^{t} g_2(t,j,k)u(j)u(k) + \cdots$$

where

$$y_0(t) = F(t,0,\ldots,0), \quad g_1(t,j) = \partial F/\partial u(j), \quad g_2(t,j,k) = \frac{1}{2}\frac{\partial^2 F}{\partial u(j)\partial u(k)}$$

Discrete time **Volterra series**.

Continuous time Volterra series by analogy

$$y(t) = y_0(t) + \int_{-\infty}^{\infty} h_1(t,\sigma)u(\sigma)d\sigma + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t,\sigma_1,\sigma_2)u(\sigma_1)u(\sigma_2)d\sigma_1 d\sigma_2 + \cdots$$

Time invariant, $y_o = 0$:

$$y(t) = \int_{-\infty}^{\infty} h_1(t-\sigma)u(\sigma)d\sigma + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t-\sigma_1, t-\sigma_2)u(\sigma_1)u(\sigma_2)d\sigma_1 d\sigma_2 + \cdots =$$

$$\int_{-\infty}^{\infty} h_1(\sigma)u(t-\sigma)d\sigma + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\sigma_1,\sigma_2)u(t-\sigma_1)u(t-\sigma_2)d\sigma_1 d\sigma_2 + \cdots$$

Continuous time **Volterra series**. h_1, h_2,...: **kernels**

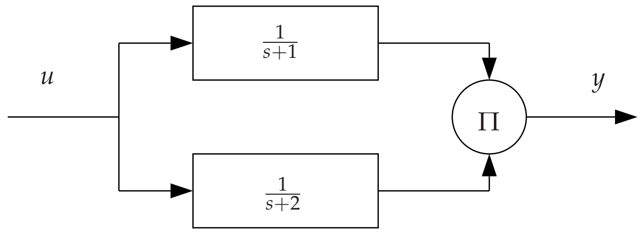
Transfer functions

$$H_1(s) = \int_{-\infty}^{\infty} h_1(\sigma)e^{-s\sigma}\,d\sigma$$

$$H_2(s_1,s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\sigma_1,\sigma_2)e^{-s_1\sigma_1 - s_2\sigma_2}\;d\sigma_1\,d\sigma_2$$

etc.

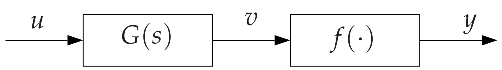
Example: an easy case



$$h_2(t_1,t_2) = e^{-(t_1+2t_2)}\Delta(t_1)\Delta(t_2)$$

$$H_2(s_1,s_2) = \frac{1}{(s_1+1)(s_2+2)}$$

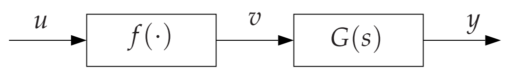
Also easy: Wiener system



If $f(v) = \sum a_j v^j$ then the transfer functions are

$$H_1(s) = a_1G(s), \quad H_2(s_1,s_2) = a_2G(s_1)G(s_2),$$
$$H_3(s_1,s_2,s_3) = a_3G(s_1)G(s_2)G(s_3), \ldots$$

Also easy: Hammerstein system



If $f(u) = \sum a_j u^j$ then the transfer functions are

$$H_1(s) = a_1G(s), \quad H_2(s_1,s_2) = a_2G(s_1+s_2),$$
$$H_3(s_1,s_2,s_3) = a_3G(s_1+s_2+s_3), \ldots$$

Response to exponentials

System: $H_2(s_1, s_2)$

Input: $u(t) = \beta_1 e^{a_1 t} + \beta_2 e^{a_2 t}$

Output:

$$y(t) = \beta_1^2 H_2(a_1, a_1)e^{2a_1 t} + + \beta_1\beta_2 \left(H_2(a_1, a_2) + H_2(a_2, a_1)\right) e^{(a_1+a_2)t} + \\ + \beta_2^2 H_2(a_2, a_2)e^{2a_2 t}$$

- New exponentials created
- No superposition

Simple nonlinear frequency analysis

System: $H_2(s_1, s_2)$

Input: $u(t) = 2A\cos\omega t = Ae^{i\omega t} + Ae^{-i\omega t}$

Output:

$$y(t) = A^2 \left(|H_2(i\omega, i\omega)| \cos(2\omega t + \phi) + H_2(i\omega, -i\omega) + H_2(-i\omega, i\omega)\right)$$

$$\phi = \arg H_2(i\omega, i\omega)$$

- New frequencies created

Important property

- $h(t_1, t_2)$ and $h(t_2, t_1)$ give the same input-output relation.
- Then $h_{sym}(t_1, t_2) = \frac{1}{2}(h(t_1, t_2) + h(t_2, t_1))$ also gives the same input-output relation.
- $h_{sym}(t_1, t_2)$ is called the *symmetric kernel* since obviously $h_{sym}(t_1, t_2) = h_{sym}(t_2, t_1)$
- $H_{sym}(s_1, s_2) = H_{sym}(s_2, s_1)$ if H_{sym} is the Laplace transform of h_{sym}.
- These properties extend to kernels of higher order.

Bilinear systems – kernels

$$\dot{x} = Ax + Dx\,u + bu, \quad y = cx, \quad x(0) = 0$$

the input-output relation is the Volterra series

$$y(t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{n,tri}(t_1, \ldots, t_n) u(t - t_1) \cdots u(t - t_n)\, dt_1 \ldots dt_n$$

where the kernels are given by

$$h_{n,tri}(t_1, \ldots, t_n) = ce^{At_n} De^{A(t_{n-1}-t_n)} D \cdots De^{A(t_1-t_2)} b, \\ \text{if } t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$$

and $h_{n,tri}(t_1, \ldots, t_n) = 0$ otherwise. *Triangular kernels*

Bilinear systems – transfer functions

The n-th order transfer function is

$$H_n(s_1, \dots, s_n) = c(\sigma_n I - A)^{-1} D (\sigma_{n-1} I - A)^{-1} D \cdots \cdots D(\sigma_1 I - A)^{-1} b$$

where

$$\begin{aligned} \sigma_1 &= s_1 \\ \sigma_2 &= s_1 + s_2 \\ &\vdots \\ \sigma_n &= s_1 + s_2 + \cdots + s_n \end{aligned}$$

General nonlinear systems

For a more general nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

the Volterra series can be calculated from the *Carleman bilinearization*.

A Carleman bilinearization of order n is a bilinear system whose first n kernels agree with the corresponding kernels of the original system.

Slightly nonlinear DC-motor

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + \epsilon x_2^2 + u \end{aligned}$$

Carleman (bi)linearization of second order:

$$\frac{d}{dt} X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} X + u \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} X$$

where

$$X = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{bmatrix}^T$$

Slightly nonlinear DC-motor, cont'd.

The first two transfer functions of the infinite Volterra series are:

$$H_1(s) = c(sI - A)^{-1} b = \frac{1}{s(s+1)}$$

$$H_2(s_1, s_2) = c((s_1 + s_2)I - A)^{-1} D (s_1 I - A)^{-1} b = \frac{2\epsilon}{(s_1 + s_2 + 2)(s_1 + s_2 + 1)(s_1 + s_2)(s_1 + 1)}$$

Volterra series transfer functions - $+$

- Extends frequency domain techniques to nonlinear systems.
- Easy to compute for linear systems and static nonlinearities in simple connections.
- Can be used to estimate distortion and out-of-band signals in communication systems.
- Can be the basis of design methods.
- Connects to nonlinear identification methods (Schoukens et al.)
- Algorithms exist for most problems.

Volterra series transfer functions - $-$

- High computational complexity in the general case.
- Feedback often increases complexity.
- The infinite series is only guaranteed to converge locally.
- Truncation errors are difficult to estimate.