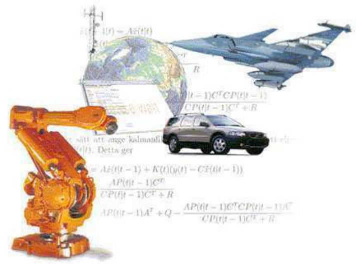
Dynamic Systems

Lecture 2 Observability



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Observability

Continuous time model:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t), u(t))$$

Discrete time model:

$$x(t+1) = f(x(t), u(t)), \quad y(t) = h(x(t))$$

Can you compute the state x from the output y and the input u?

Observability, discrete time:

Look at several points in time to extract more information

$$\begin{aligned}
y(t) &= h(x(t)) \\
y(t+1) &= h(x(t+1)) = h(f(x(t), u(t)) = h^{(1)}(x(t), u(t)) \\
y(t+2) &= h^{(1)}(x(t+1), u(t+1)) = h^{(1)}(f(x(t), u(t)), u(t+1)) = \\
&\quad h^{(2)}(x(t), u(t), u(t+1)) \\
&\vdots \\
yt+N) &= h^{(N)}(x(t), u(t), u(t+1), \ldots, u(t+N-1))
\end{aligned}$$

This is a system of nonlinear equations in $x(t)$. Solvability implies observability.

Observability, continuous time:

Differentiate the output to extract more information

$$\begin{aligned}
y &= h(x) \\
\dot{y} &= h_x(x)\dot{x} = h_x(x)f(x,u) = h^{(1)}(x,u) \\
\ddot{y} &= h_x^{(1)}(x,u)\dot{x} + h_u^{(1)}(x,u)\dot{u} = h_x^{(1)}(x,u)f(x,u) + h_u^{(1)}(x,u)\dot{u} = \\
&= h^{(2)}(x,u,\dot{u}) \\
&\vdots \\
y^{(N)} &= h^{(N)}(x, u, \dot{u}, \ldots, u^{(N-1)})
\end{aligned}$$

This is a system of nonlinear equations in x. Solvability implies observability.

No input. Alternative description

Define

$$L_f = \sum f_i(x) \frac{\partial}{\partial x_i}$$

Then

$$\begin{aligned} y &= h(x) \\ \dot{y} &= (L_f h)(x) \\ \ddot{y} &= (L_f^2 h)(x) \\ &\vdots \\ y^{(N)} &= (L_f^N h)(x) \end{aligned}$$

Solvability usually depends on u

Example:

$$\dot{x}_1 = x_2 u, \quad \dot{x}_2 = x_1 x_2, \quad y = x_1$$

The system of equations

$$\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 u \end{aligned}$$

can not be solved for x_2 if $u = 0$.

Mathematics of equation solving

- Linear equations: solvability determined by rank test
- Polynomial equations: extensive (and difficult) mathematical theory (ideals, Gröbner bases, characteristic sets, elimination theory, cylindrical algebraic decomposition). Very high computational complexity.
- Local properties of general equations: Implicit function theorem.

Implicit function theorem

Consider the equation

$$f(x, y) = 0$$

where the dimensions of x and f are equal (same number of unknowns and equations).

Assume that

$$f(x_o, y_o) = 0, \quad f_x(x_o, y_o) \text{ nonsingular}$$

(f_x is the Jacobian of f with respect to x)
Then, for all y close to y_o the equation has a solution x which is locally unique.

Local observability

Compute the Jacobian

$$J = \begin{bmatrix} h_x(x) \\ h_x^{(1)}(x,u,\dot{u}) \\ \vdots \\ h_x^{(N)}(x,u,\dot{u},\dots,u^{(N-1)}) \end{bmatrix}$$

J full rank at $x_0 \Rightarrow x$ is uniquely determined by u, y and their derivatives in a neighborhood of x_0 (implicit function theorem)

Linear time invariant systems

$$\dot{x} = Ax + Bu, \quad y = Cx$$

The Jacobian J is constant and has the form

$$J = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}$$

- J full rank $\Rightarrow x$ can be solved uniquely (globally)
- This is the classical observability test for linear systems
- Cayley-Hamilton $\Rightarrow$ no need to take $N > n$.

Linear time-varying systems

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x$$

The Jacobian J has the form

$$J = \begin{bmatrix} C \\ \frac{\partial}{\partial t}C + CA \\ \vdots \end{bmatrix}$$

See the exercises.

Quantitative measure of observability

$$\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t)$$

The energy in the output is given by

$$\int_{t_0}^{t_1} y(t)^T y(t)\, dt = x(t_0)^T M(t_0,t_1) x(t_0)$$

where M is the *Observability Gramian*:

$$M(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(t,t_0) C^T(t) C(t) \Phi(t,t_0)\, dt$$

Observability and the Gramian

From the definition of the observability Gramian:

$$\int_{t_0}^{t_1} \Phi^T(t,t_0)C^T(t)y(t)\,dt = M(t_0,t_1)x(t_0)$$

Theorem:
observability on $[t_0,t_1] \Leftrightarrow M(t_0,t_1)$ nonsingular

Note: This is a smoothing estimate of $x(t_0)$ based on $y(t)$, $t_0 \le t \le t_1$.

Observability, discrete time

Iterating the state equation with $u = 0$ gives

$$\begin{bmatrix} y(t_0) \\ y(t_0+1) \\ \dots \\ y(t_1-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C(t_0) \\ C(t_0+1)A(t_0) \\ \vdots \\ C(t_1-1)A(t_1-2)\cdots A(t_0) \end{bmatrix}}_{O(t_0,t_1)} x(t_0)$$

$x(t_0)$ can be computed from $y(t_0),\dots,y(t_1-1)$ if $O(t_0,t_1)$ has full rank.
$O^T O$ is the discrete time observability Gramian.

For constant A and C one can define

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

in both continuous and discrete time.

Theorem
The range and null spaces of $M(t_0,t_1)$
coincide with
the range and null spaces of $\mathcal{O}^T\mathcal{O}$
for all $t_1 > t_0$.

The corresponding discrete time result is trivial.

Change of state variables

$$x = T(z), \quad T \text{ one-to-one} \quad T_z \text{ nonsingular}$$

gives

$$\dot{z} = T_z^{-1} f(T(z),u) = \tilde{f}(z,u), \quad y = h(T(z)) = \tilde{h}(z)$$

To test observability, we have to compute

$$\tilde{h}^{(1)}(z,u) = \tilde{h}_z(z)\tilde{f}(z,u) = h_x(T(z))T_z T_z^{-1} f(T(z),u) = h^{(1)}(T(z),u)$$

In general $\tilde{h}^{(j)}(z,u,\dot{u},\dots,u^{(j-1)}) = h^{(j)}(T(z),u,\dot{u},\dots,u^{(j-1)})$

Change of state variables, cont'd

The Jacobian is

$$\tilde{J} = \begin{bmatrix} \tilde{h}_z(z) \\ \tilde{h}_z^{(1)}(z,u,\dot{u}) \\ \vdots \\ \tilde{h}_z^{(N)}(z,u,\dot{u},\ldots,u^{(N-1)}) \end{bmatrix} = \begin{bmatrix} h_x(T(z))T_z \\ h_x^{(1)}(T(z),u,\dot{u})T_z \\ \vdots \\ h_x^{(N)}(T(z),u,\dot{u},\ldots,u^{(N-1)})T_z \end{bmatrix} = JT_z$$

Since T_z is nonsingular, the rank of J is the same in both coordinate systems.

Change of state variables, linear systems

$$x = Tz, \quad T \text{ nonsingular matrix}$$

gives

$$\dot{z} = \underbrace{T^{-1}AT}_{\tilde{A}}z + \underbrace{T^{-1}B}_{\tilde{B}}u, \quad y = \underbrace{CT}_{\tilde{C}}z$$

$$\tilde{\mathcal{O}} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \end{bmatrix} = \begin{bmatrix} CT \\ CTT^{-1}AT \\ \vdots \end{bmatrix} = \mathcal{O}T$$

Unobservability and observers

Consider the systems

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \qquad \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

- Both systems are unobservable
- One of them can be given a converging observer
- For the other one this is not possible
- Why?

Canonical form for observability

$$\dot{z} = \underbrace{T^{-1}AT}_{\tilde{A}}z + \underbrace{T^{-1}B}_{\tilde{B}}u, \quad y = \underbrace{CT}_{\tilde{C}}z$$

Theorem Let the rank of $\mathcal{O}$ be r. Then T can be chosen so that

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & 0 \end{pmatrix}$$

where $\tilde{A}_{11}$ is $r \times r$, $\tilde{C}_{11}$ is $p \times r$ and $\tilde{A}_{11}, \tilde{C}_{11}$ observable. *Non-unique !*

Note: The partition of eigenvalues between $\tilde{A}_{11}$ and $\tilde{A}_{22}$ is unique: it makes sense to speak of "observable" and "unobservable" eigenvalues.

Similar theorem for time-varying systems, see Rugh.

PBH tests

Popov 1966, Belevitch 1968, Hautus 1969:
Theorem: A, C observable if and only if

$$Ap = \lambda p, \; Cp = 0 \Rightarrow p = 0$$

Theorem: A, C observable if and only if

$$\text{rank} \begin{pmatrix} C \\ sI - A \end{pmatrix} = n$$

for all complex s.

Detectability

Detectable system: All eigenvalues of the unobservable part lie strictly in left half plane.

The following are equivalent:

- The system is detectable.
- It is possible to choose the observer gain K so that $A - KC$ has all its eigenvalues strictly in the left half plane.

Proof: PBH and canonical form.

More on nonlinear observability

The definition of nonlinear observability is tricky. Is it for instance reasonable to say that the following system is observable?

$$\dot{x} = 1$$

$$y = \begin{cases} 0 & x < 10^{100} \\ (x - 10^{100})^2 & x \geq 10^{100} \end{cases}$$

Some terminology

$$\dot{x} = f(x, u), \quad y = h(x), \quad x(0) = x_o$$

has solution $x(t) = \pi(t, x_o, u)$
x_1, x_2 **indistinguishable**:
$h(\pi(t; x_1, u)) = h(\pi(t; x_2, u))$, all $t \geq 0$, all u
$I(x)$ = all points indistinguishable from x

- system **observable at** x_0: $I(x_0) = \{x_0\}$
- system **observable**: $I(x_0) = \{x_0\}$ all x_0

More terminology

Often sufficient to distinguish points that are close:

- system **weakly observable** at x_0: exists neighborhood (nbh) U such that $I(x_0) \cap U = \{x_0\}$

Often necessary to distinguish points without moving to far:

- x_1, x_2 **U-indistinguishable** if they are distinguishable as long as both trajectories lie entirely in U.
- $I_U(x) =$ all points U-indistinguishable from x

Local weak observability

- System **locally observable** at x_0: $I_U(x_0) = \{x_0\}$ every open nbh U of x_0.
- System **locally weakly observable** at x_0: Exists open nbh U of x_0 such that $I_V(x_0) = \{x_0\}$ for every open nbh V of x_0 with $V \subset U$
- System **locally weakly observable**: locally weakly observable at every x_0;
  "x can instantaneously be distinguished from its neighbors"

Relationships:

$$\begin{array}{ccc} \text{locally observable} & \Rightarrow & \text{observable} \\ \Downarrow & & \Downarrow \\ \text{locally weakly observable} & \Rightarrow & \text{weakly observable} \end{array}$$

For a *linear* system they are all equivalent.

Lie derivative evaluation (piecewise constant u)

Define $f_1(x) = f(x, u_1)$

$$(L_{f_1} h)(x) = h_x(x) f_1(x) = \dot{y}$$

The *Lie-derivative* in the direction f_1.

More generally, if $f = f_1$ for t_1 units of time, $f = f_2$ for t_2 units of time,...

$$\left(\frac{\partial^k}{\partial t_1 \cdots \partial t_k} y(t_1 + t_2 + \cdots + t_k) \right) \Big|_{t_1 = \cdots = t_k = 0} = \left(L_{f_1} L_{f_2} \ldots L_{f_k} h \right)(x_0)$$

Test for local weak observability

Consider all elements of the form

$$h_x,\ (L_{f_1} h)_x,\ (L_{f_2} h)_x,\ \ldots,\ (L_{f_1} L_{f_2} h)_x,\ \ldots,\ (L_{f_1} L_{f_2} \cdots L_{f_k} h)_x,\ \ldots$$

for all possible choices of u.

- Observability rank condition at x_0: n linearly independent rows among these elements
- Hermann and Krener 1977:
  - Observability rank condition at $x_0 \Rightarrow$ local weak observability at x_0
  - Local weak observability for all $x \Rightarrow$ observability rank condition generic (satisfied on open dense subset)