Chasles Theorem

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Chasles theorem is one of the most fundamental results in kinematics.

Theorem 1 (Chasles) *Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.*

Proof 1 *Consider a general* 4×4 *homogeneous transformation matrix*

$$A = \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix} \tag{1}$$

In order to continue we will now change bases in order reveal the structure. This can be done by perform a similarity transform of the A*-matrix according to*

$$\Lambda = \begin{pmatrix} Q^T & -Q^T c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q^T RQ & Q^T Rc - Q^T c + Q^T d \\ 0 & 1 \end{pmatrix} \tag{2}$$

Let us start investigating the rotation part, i.e., the upper left 3×3 *sub-matrix of* Λ*. The* Q *matrix can now be chosen according to*

$$Q = \begin{pmatrix} v_1 & v_2 & u \end{pmatrix} \tag{3}$$

where u *is the eigenvalue of* R *corresponding to eigenvalue* 1 *(more specifically it is the axis of rotation). The other two vectors* v_1 *and* v_2 *are chosen so that they together with* u *form a real basis. This implies that the* 3×3 *upper left part* $(Q^T RQ)$ *of* Λ *is reduced to a rotation about the* z *axis according to*

$$Q^T RQ = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

When it comes to the translation, we have that

$$Q^T Rc - Q^T c + Q^T d = (Q^T RQ - I)Q^T c + Q^T d. \tag{5}$$

Let us now define

$$\bar{c} = Q^T c, \tag{6a}$$

$$\bar{d} = Q^T d, \tag{6b}$$

$$\tag{6c}$$

allowing us to write (5) *according to*

$$\begin{pmatrix} \cos\varphi - 1 & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{c}_x \\ \bar{c}_y \\ \bar{c}_z \end{pmatrix} + \begin{pmatrix} \bar{d}_x \\ \bar{d}_y \\ \bar{d}_z \end{pmatrix} \tag{7}$$

If the top 2×2 *matrix of* $(Q^T RQ - I)$ *is nonsingular we can solve the first two equations of*

$$(Q^T RQ - I)\bar{c} = -\bar{d} \tag{8}$$

for $\bar{c}_x$ *and* $\bar{c}_y$ *and, without loss of generality, let* $\bar{c}_z = 0$*. In that case we have* Λ *in the form*

$$\Lambda = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

where k *is given by the third component in*

$$\bar{d} = Q^T d. \tag{10}$$

Hence, the rigid body motion is described by a rotation about the z*-axis through an angle* φ *followed by a translation along the* z*-axis through a distance* k*.*

If the top 2×2 *submatrix of* $(Q^T RQ - I)$ *is singular, then* $Q^T RQ = I$*. This means that* Λ *is a pure translation.*