Lecture 1 – Rigid Body Motion

Mikael Norrlöf and Thomas Schön,
Division of Automatic Control,
Department of Electrical Engineering,
Linköping University.
Email: {mino,schon}@isy.liu.se

Content

- Rigid body transformation
- Rotation
  - Rotation matrices
  - Euler’s theorem
  - Parameterization of SO(3)
- Homogeneous representation
  - Matrix representation
  - Chasles’ theorem

Background to modeling

Kinematics

- studies the motion of objects without consideration of the circumstances leading to the motion

Dynamics

- studies the relationship between the motion of objects and its causes

Rigid body motion

The motion of a rigid body can be parameterized as
- position
- orientation
of one point of the object. The configuration.
The motion of a rigid body can be parameterized as
- position - orientation
of one point of the object. The **configuration**.

**Content**

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**Representation of orientation**

- Angle – axis representation
- Euler angles
- Quaternion
- Exponential coordinates
- ...

**Euler angles**
Euler angles

The order of rotation axes is important

\[
R_1(\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\alpha) & \sin(\alpha) \\
0 & -\sin(\alpha) & \cos(\alpha)
\end{bmatrix}
\]

\[
R_2(\alpha) = \begin{bmatrix}
\cos(\alpha) & 0 & -\sin(\alpha) \\
0 & 1 & 0 \\
\sin(\alpha) & 0 & \cos(\alpha)
\end{bmatrix}
\]

\[
R_3(\alpha) = \begin{bmatrix}
\cos(\alpha) & \sin(\alpha) & 0 \\
-\sin(\alpha) & \cos(\alpha) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Gimbal lock (Apollo IMU Gimbal lock 1, 2)

\[
R(\alpha, \beta, \gamma) = \begin{bmatrix}
0 & \cos \gamma \sin \alpha - \cos \alpha \sin \gamma & \cos \alpha \cos \gamma + \sin \alpha \sin \gamma & 0 \\
0 & \cos \alpha \cos \gamma + \sin \alpha \sin \gamma & \cos \alpha \sin \gamma - \cos \gamma \sin \alpha & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Quaternions

Sir William Rowan Hamilton (1809-1865)

Lectures on Quaternions: Containing a Systematic Statement of
A New Mathematical Method

Of which the principles were communicated in 1843 to the Royal Irish Academy; and which has since formed the subject of successive courses of lectures, delivered in 1848 and subsequent years in the halls of Trinity College, Dublin: with numerous illustrative diagrams, and with some geometrical and physical applications.
**Quaternions**

Generalization of complex numbers to 3D.

\[ s + i x + j y + k z \]

with \( i^2 = j^2 = k^2 = i j k = -1, i j = -j i = k, j k = -k j = i, k i = -i k = j \).

A quaternion is usually represented as \( q = s + v \) with

- \( s \) scalar (real part)
- \( v \) vector in \( \mathbb{R}^3 \) (complex part)

Unit quaternion \( ||q|| = 1 \).

**Rotation with quaternions**

Angle axis to quaternion

\[ q = \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \cdot v \]

Composition of rotations, \( q_1 \) then \( q_2 \)

\[ q = q_2 q_1 \]

Rotation of a vector, \( u = R v \)

\[ v_q = <0, v> \text{, } q \text{ is quaternion representation of } R \]

\[ u_q = q v q^{-1} = <0, u> \]

\[ R_q(q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2 q_1 q_2 - 2 q_0 q_3 & 2 q_1 q_3 + 2 q_0 q_2 \\ 2 q_1 q_2 + 2 q_0 q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2 q_2 q_3 - 2 q_0 q_1 \\ 2 q_1 q_3 - 2 q_0 q_2 & 2 q_2 q_3 + 2 q_0 q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \]

**Some remarks**

- \( q \) and \(-q\) represent the same rotation

- \( q = <s,v> \text{ and } q^{-1} = <s,-v> \)
Quaternions

- Can only represent orientation
- Quaternion math is not so well known
- Compact representation, based upon angle axis rep.
- Simple interpolation methods
- No gimbal lock
- Simple composition
- Linear (bi-linear) dynamics, (NASA)

Comparison for different operations

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<th># multiplies</th>
<th># add/subtracts</th>
<th>Total operations</th>
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<th>Storage</th>
<th># multiplies</th>
<th># sin/cos</th>
<th>Total operations</th>
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Illustration of Euler’s Theorem

Hence, the effect of the rotation $R$ is to rotate vectors in the plane spanned by $v_1$ and $v_2$ through an angle $\varphi$ along $u$. This shows that $R$ rotates a rigid body about $u$ through an angle $\varphi$. This concludes the proof of Euler’s theorem.

Canonical Representation of the Rotation Matrix

There is a canonical representation of any rotation matrix $R$, that allows us to view it as a rotation through an angle $\varphi$ about the z-axis.

Define the orthonormal matrix $Q = \begin{pmatrix} v_1 & v_2 & u \end{pmatrix}$ and

$$
\Lambda = \begin{pmatrix} 
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1 
\end{pmatrix}
$$

Then we can show that

$$
R = QAQ^T
$$

Recall that “change of basis = similarity transformation”
Homogeneous Representation

How do we represent rigid body motion in general, i.e., both orientation and translation.

A full rigid-body motion is denoted by \( g = (R, T) \)

The set of all possible configurations of a rigid body can be described by the space of rigid-body motions or special Euclidean transformations

\[
SE(3) \triangleq \{ g = (R, T) | R \in SO(3), T \in \mathbb{R}^3 \}
\]

Homogeneous Representation

The equation

\[
X^w = R^{wc} X^c + T^{wc}
\]

is affine, we would like to get rid of the additive term.

We can convert the affine transformation into a linear transformation by augmenting an additional 1 to \( X \)

\[
\bar{X} = \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{pmatrix}
\]

Homogeneous Representation

What is linear about this?

Let us have a look

\[
\bar{X}^w = \begin{pmatrix} X^w \\ 1 \end{pmatrix} = \begin{pmatrix} R^{wc} & T^{wc} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^c \\ 1 \end{pmatrix} = \bar{g}^{wc} \bar{X}^c
\]

Chasle’s Theorem

Proof:

Consider a general 4x4 homogeneous matrix (describing a rigid body motion)

\[
A = \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}
\]

We will now change basis in order to see better (again, recall that “change of basis = similarity transform”).

Perform a similarity transformation of the matrix A

\[
\Lambda = \begin{pmatrix} Q^T & -Q^T c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q^T R Q & Q^T R c - Q^T c + Q^T d \\ 0 & 1 \end{pmatrix}
\]
Chasle's Theorem

Proof (continued):

Rotation:

Recall that $v_1$, $v_2$ and $u$ are orthogonal

Choose $Q$ according to $Q = \begin{pmatrix} v_1 & v_2 & u \end{pmatrix}$

$$Q^T R Q = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a rotation about the $z$-axis

Hence, the rigid body motion is described by a rotation about the $z$-axis through an angle $\varphi$ followed by a translation along the $z$-axis through a distance $k$.

If the top $2x2$ submatrix of $(Q^T R Q - I)$ is singular, then $Q^T R Q = I$. This means that $\Lambda$ is a pure translation.

The proof is finished.

Chasle's Theorem – Screw Motion

The motion implied by Chasle's theorem is like when you screw in that it rotates and translates along the same axis.
Further Studies Besides Course Literature

- R.M. Murray, Z. Li, and S.S. Sastry: *A mathematical introduction to Robotic Manipulation* (Chapter 2)
- James Diebel: *Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors*
- Erik B. Dam, Martin Koch, and Martin Lillholm: *Quaternions, Interpolation and Animation*