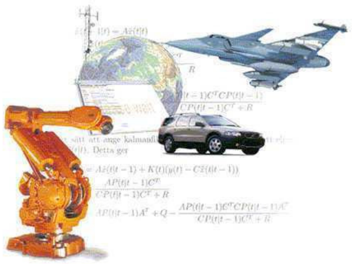
Dynamic Systems

Lecture 5. Realizability and Realizations



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Input-output relations

State space description:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t)$$

$$x(t_0) = 0 :, \quad y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

Impulse response:

$$h(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$$

A, B, C constant:

$$h(t,\tau) = Ce^{A(t-\tau)}B = h(t-\tau,0), \quad G(s) = C(sI-A)^{-1}B$$

Fact: $h(t,\tau)$ unaffected by variable change $x(t) = P(t)z(t)$

Realization

- The *realization* problem is to find a state space description for a given input-output relation.
- A *minimal realization* has the lowest possible dimension of the state space.

Finite dimensional realizations

Is there a finite-dimensional linear system that has the impulse response

$$h(t,\tau) = \frac{e^{-\frac{1}{4(t-\tau)}}}{2\sqrt{\pi}(t-\tau)^{1.5}} \ ?$$

(heat conduction in a rod)

Theorem An impulse response has a finite dimensional realization if and only if it can be factorized as

$$h(t,\tau) = M(t)N(\tau)$$

The simplest realization has the form

$$\dot{x}(t) = N(t)u(t), \quad y(t) = M(t)x(t)$$

Minimality, controllability and observability

With the variable change $x = \Phi(t, t_0)z$ the system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x$$

is transformed into

$$\dot{z} = \Phi(t_0, t)B(t)u, \quad y = C(t)\Phi(t, t_0)z$$

with the same impulse response, and the same ranks of the Gramians.

Theorem A realization is minimal if and only if, for some $t_0 < t_1$, both $W(t_0, t_1)$ and $M(t_0, t_1)$ (the controllability and observability Gramians) are nonsingular.

Realization of time invariant systems

Fact A, B, C constant $\Rightarrow$ h is continuously differentiable with $h(t, \tau) = h(t - \tau, 0)$

Theorem
- h is continuously differentiable
- $h(t, \tau) = h(t - \tau, 0)$
- h has a finite realization

$\Rightarrow$

h has a *minimal* realization with constant A, B, C.

Time-invariant case. Cont'd.

Easier to consider $G(s)$ than $h(t)$.

Example Does $G(s) = e^{-\sqrt{s}}$ have a finite-dimensional realization? (heat conduction in a rod)
Theorem $G(s)$ has time-invariant finite-dimensional realization $\Leftrightarrow$ each element is a strictly proper rational function

A simple time invariant realization of $G(s)$

$$d(s) = s^r + d_{r-1}s^{r-1} + \cdots + d_1 s + d_0$$

is the least common multiple of the denominator polynomials.

$$d(s)G(s) = N_{r-1}s^{r-1} + \cdots + N_1 s + N_0$$

The N_i are constant $p \times m$ matrices.
A controllable realization (usually *not* minimal)

$$A = \begin{bmatrix} 0_m & I_m & \dots & 0_m \\ \vdots & & \ddots & \vdots \\ 0_m & 0_m & \dots & I_m \\ -d_0 I_m & -d_1 I_m & \dots & -d_{r-1} I_m \end{bmatrix}, \quad B = \begin{bmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{bmatrix}$$

$$C = \begin{bmatrix} N_0 & N_1 & \dots & N_{r-1} \end{bmatrix}$$

Example of non-minimal realization

$$G(s) = \begin{bmatrix} \frac{1}{2+3s+s^2} & \frac{3+s}{2+3s+s^2} \\ \frac{1}{2+s} & \frac{1}{2+s} \end{bmatrix} = \frac{1}{2+3s+s^2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\right)$$

gives

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Example cont'd. Observability decomposition

Decomposing into obervable and unobservable parts:

$$A_o = \begin{bmatrix} -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & -3 & -2 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Minimal realization (observable part)

$$A_{min} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B_{min} = \begin{bmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad C_{min} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Markov parameters

Continuous time impulse response

$$h(t) = Ce^{At}B = CB + tCAB + \frac{t^2}{2}CA^2B + \cdots$$

Discrete time input-output relation

$$y(t) = CBu(t-1) + CABu(t-2) + CA^2Bu(t-3) + \cdots$$

$h_j = CA^{j-1}B$ are called *Markov parameters*.

Realization $\Leftrightarrow$ find A, B, C, given the Markov parameters.

A useful relation

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{j-1}B \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2 & \dots & h_j \\ h_2 & h_3 & \dots & h_{j+1} \\ \vdots & & & \vdots \\ h_i & h_{i+1} & \dots & h_{i+j} \end{bmatrix}}_{H_{ij}}$$

Finite dimensional realizations

Theorem The impulse response has a finite dimensional realization
$\Leftrightarrow$
There exist finite values of i and j for which the rank of H_{ij} attains its maximal value. (This maximal value is the dimension of the minimal realization.)

Theorem If A_1, B_1, C_1 and A_2, B_2, C_2 are both minimal realizations of the same input-output relation, then there is a matrix T such that

$$A_2 = T^{-1}A_1T, \quad B_2 = T^{-1}B_1, \quad C_2 = C_1T$$