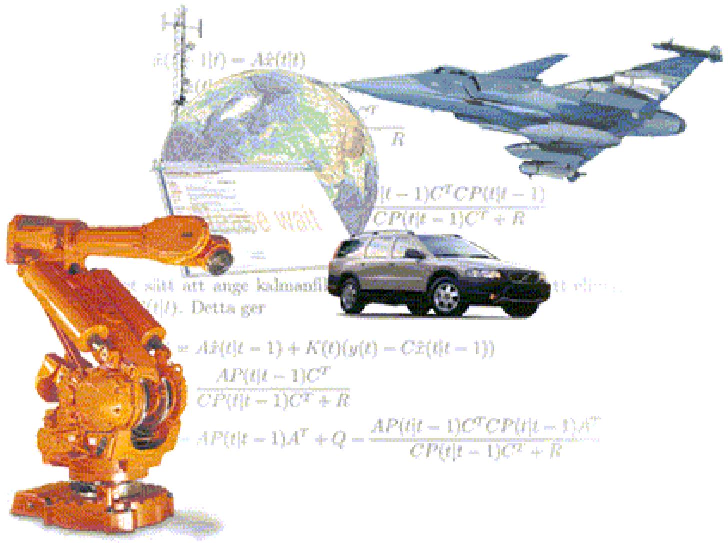
Parametric identifiability from parameter-free equations



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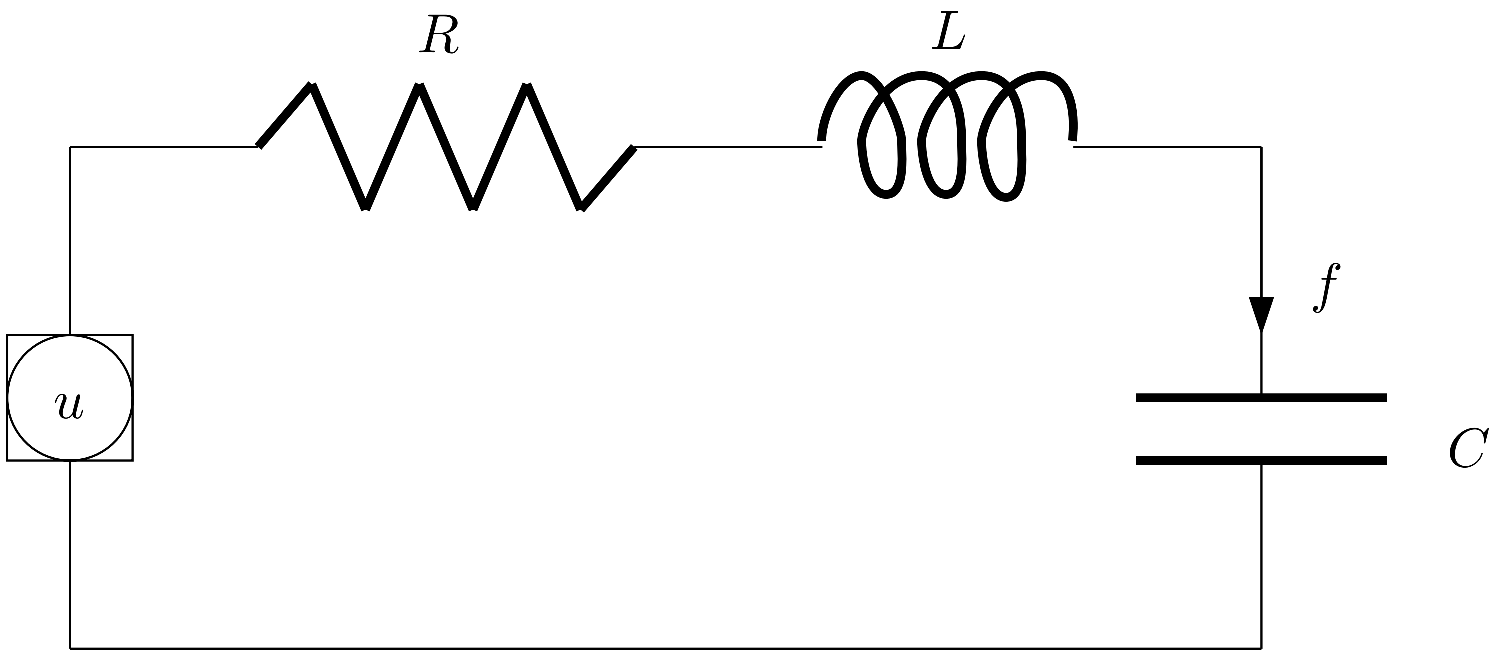
Main idea

If you want to determine identifiability:

Look at the equations where the parameters are absent.

A simple example

Series connection of a resistor, an inductor and a capacitor



If the voltage u and the current f are measured, can you determine the resistance, the inductance and the capacitance?

Mathematical model

With $f =$ current, e_L, e_C, e_R voltages, the model is

$$\begin{aligned} L\dot{f} &= e_L \\ C\dot{e}_C &= f \\ e_R &= Rf^3 \\ e_L + e_R + e_C &= u \end{aligned}$$

Parameters to identify: L, C, R

- General procedure (Ljung, Glad 1994): Use elimination theory to get a relation containing only L (C, R), f and u (and their derivatives). This relation tells all about identifiability.
- Completely algorithmic for polynomial models.
- High complexity in general.

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Using model structure

- Most engineering models are formed from simple components
  - Electric circuits
  - Hydraulic systems
  - Rotating machinery
  - and so on
- Each component has usually a simple mathematical description
- There is an interconnection structure
  - Kirchhoff's laws
  - Bond graphs
- Modern modeling tools – e.g. Modelica – use this structure

Identifiability of an inductor

$$L\dot{f} = e, \quad \dot{L} = 0$$

- L inductance (parameter to be identified)
- e voltage, f current
- If e and f are measured L is trivially identifiable

Running the identifiability algorithm

$$p_1 = L\dot{f} - e, \quad p_2 = \dot{L}$$

$$p_3 = \dot{f}^2 p_2 - \dot{f}\dot{p}_1 + \ddot{f} p_1 = -e\ddot{f} + \dot{f}\dot{e}$$

There are now two equivalent sets of equations (with an inequation)

$$\begin{cases} L\dot{f} - e = 0 \\ \dot{L} = 0 \\ \dot{f} \neq 0 \end{cases} \Leftrightarrow \begin{cases} L\dot{f} - e = 0 \\ -e\ddot{f} + \dot{f}\dot{e} = 0 \\ \dot{f} \neq 0 \end{cases}$$

- The middle equation to the right does not contain L
- It is the key to identifiability when the inductor is part of a larger network.

The linear resistor

With similar calculations a linear resistor has the description

$$e = Rf$$
$$\dot{e}f = e\dot{f}$$

where e is voltage, f is current and R is resistance.

What happens if an inductor (voltage e_1) is connected in series with a resistor (voltage e_2)? f is common current, e voltage across both components.

$$
\begin{aligned}
0 &= p_1 = e_1 + e_2 - e \\
0 &= p_2 = e_1 \ddot{f} - \dot{f} \dot{e}_1 \\
0 &= p_3 = \dot{e}_2 f - e_2 \dot{f}
\end{aligned}
\tag{1}
$$

- e and f are measured
- e_1 and e_2 are *not* measured
- can they be calculated? (implies that R and L can be calculated)

Some simple calculations

Forming

$$p_4 = \dot{f} p_3 - \dot{f} f \dot{p}_1 + \dot{f}^2 p_1 - f p_2 = \dot{f} f \dot{e} - \dot{f}^2 e + (\dot{f}^2 - f \ddot{f}) e_1$$

shows that e_1 can indeed be calculated from e, f (e_2 is then trivially calculated from $e_2 = e - e_1$) *provided*

$$\dot{f}^2 - f \ddot{f} \neq 0$$

This is an *excitation* condition.

The most important step still remains

The following calculation removes e_1

$$
\begin{aligned}
p_5 &= (2\dot{f}^2\ddot{f} - \dot{f}f\dddot{f} - f\ddot{f}^2)p_4 - (\dot{f}^2 - f\ddot{f})(\dot{f}\dot{p}_4 + (\dot{f}^2 - f\ddot{f})p_2) \\
&= (-\dot{f}^4 f + f^2\ddot{f}\dot{f}^2)\ddot{e} + (\dot{f}^3\ddot{f}f - \dot{f}^2 f^2\dddot{f})\dot{e} + \dot{f}^3 f\dddot{f}e - f\ddot{f}^2\dot{f}^2 e
\end{aligned}
$$

- The inductor-resistor can now be regarded as a new component having this relation between the external variables.
- R and L are completely invisible
- Nevertheless their identifiability when further connections are made can be determined from this relation.

Propagation of identifiability

- Determine for each component if the internal parameters are identifiable from the external variables.
- Determine a parameter-free relation among the external variables.
- When components are connected: determine if the external variables of the components can be determined from the external variables of the overall model.
- Determine a relation between the external variables of the overall model (free of those variables that are external to each component)
- Continue this process hierarchically as more and more complex models are connected together.
- In the process internal equations and variables are “forgotten”, reducing complexity.
- Results for subsets of connected components can be stored.

A nonlinear two-parameter model

A nonlinear resistor model

$$e = \theta_1 f + \theta_2 f^2$$

e voltage, f current.
The model is characterized by

$$\begin{aligned} 2\dot{f}^3 f e - 2\dot{f}^2 f^2 \dot{e} - \ddot{f} f^3 \dot{e} + \dot{f} f^3 \ddot{e} &= 0 \\ f\dot{e} - \dot{f} e - \theta_2 f^2 \dot{f} &= 0 \\ f\dot{e} - 2\dot{f} e + \theta_1 \dot{f} f &= 0 \end{aligned}$$

The last two equations show that θ_1 and θ_2 are identifiable provided $f\dot{f} \neq 0$.
The first equation is the parameter-free relation between the external variables.

What is the algorithm for eliminating variables?

- The basic idea is similar to Gaussian elimination, but it also handles polynomials
- It is similar to Gröbner bases, but also handles differentiation.
- The basic mathematical theory was developed by J. F. Ritt and A. Seidenberg in 1930 – 1960.
- The connection to control and systems theory was made by M. Fliess around 1990.

Ranking

- A fundamental concept is the *ranking* of variables and their derivatives.
- It tells the algorithm in what order the elimination should be done.
- A higher derivative order always ranks higher than a lower one, e.g. $\dot{y} < \ddot{y}$
- A higher power of a variable always ranks higher than a lower one, e.g. $y^3 < y^7$
- The ranking of variables is a “user choice”.

The nonlinear resistor again

For the resistor model

$$e = \theta_1 f + \theta_2 f^2$$

we could for instance use the ranking

$$f < \dot{f} < \cdots < e < \dot{e} < \cdots < \theta_1 < \dot{\theta}_1 < \cdots \theta_2 < \dot{\theta}_2 < \cdots$$

This would tell the algorithm to eliminate (if possible) first θ_2, then θ_1, then e and finally f.
In this case the elimination stops with a relation involving e and f.

The basic elimination step

- Given two multivariable polynomials p_1 and p_2 where
  - The highest derivative of y in p_1 is $y^{(m)}$.
  - The highest derivative of y in p_2 is $y^{(n)}$.
  - $n > m$
- Then $y^{(n)}$ can be eliminated by differentiating p_1 $n - m$ times (so that p_1 also contains $y^{(n)}$) and using polynomial division.
- The elimination algorithm does this again and again guided by the ranking until all desired elimination is done.
- The ranking gives a monotonicity property that ensures convergence
- Details are in the paper.

Physical analogies

- There are far-reaching analogies between different physical domains (as shown e.g. in bond-graph theory)
- The inductor-resistor model could just as well be a model of a mass moving with viscous friction.
- Identifiability calculations from one model library can in many cases be transferred directly to other libraries.

Non-polynomial models

The basic ideas also apply to non-polynomial models.

Consider for instance a general one-parameter resistor model

$$e = r(\theta, f)$$

where e is the voltage and f the current.
Identifiability of the parameter θ is guaranteed if the relation can be inverted to give a relation of the form

$$\theta = s(e, f)$$

There is then also a parameter-free relation between the external variables:

$$\dot{e} = r_f(s(e, f), f)\dot{f}$$

There is however no general finite algorithm for the overall interconnected structure.

Conclusion

Identifiability can be checked in an *hierarchical* manner using the *interconnection structure* of models.

Challenge 1

It is known that

- Classical circuit theory
- Bond graph theory
- Theory of Hamiltonian systems

are closely related.

How does the identifiability algorithm fit in?

Challenge 2

Markus Gerdin has shown that there are close connections between

- Identifiability and observability
- Solvability theory for differential-algebraic equations

How does the elimination algorithm fit in?