AUTOMATIC CONTROL
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DISCRETIZING STOCHASTIC DYNAMICAL SYSTEMS USING LYAPUNOV EQUATIONS

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Contribution

We present an algorithm using a combination of **Lyapunov equations** and analytical solutions for **discretizing** continuous-time **stochastic dynamical equations**.

Motivation

Stochastic dynamical systems are important in state estimation, system identification and control. System models are often provided in continuous time, while a major part of the applied theory is developed for discrete-time systems. Discretization of continuous-time models is hence fundamental.

Problem Formulation

Continuous-time model

$$\dot{x}(t) = Ax(t) + Bw(t)$$
$$E[w(t)w(\tau)^\mathsf{T}] = S\delta(t-\tau)$$

Discrete-time model

$$x_{k+1} = F_T x_k + w_k$$
$$E[w_k w_l^\mathsf{T}] = Q_T \delta_{kl}$$

This gives the relations

$$F_T = e^{AT}, \qquad Q_T = \int_0^T e^{A\tau} BSB^\mathsf{T} e^{A^\mathsf{T}\tau} d\tau \qquad (1)$$

Problem: How do we solve the integral (1) in a numerically good manner for arbitrary A, B, S and T?

Analytical solution

If A is nilpotent the integral (1) has an analytical solution.

Example (constant velocity model)

The system given by $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $S = 1$

results in $F_T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $Q_T = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}$.

Solution using Lyapunov Equation

Theorem *The solution to the integral* (1) *satisfies the Lyapunov equation*

$$AQ_T + Q_T A^\mathsf{T} = \underbrace{-BSB^\mathsf{T} + e^{AT}BSB^\mathsf{T}e^{A^\mathsf{T}T}}_{-V_T} \qquad (2)$$

Idea: Solve the Lyapunov equation (2) to find solution for the integral (1)!

Lemma *Eq.* (2) *has a unique solution if and only if*

$$\lambda_i(A) + \lambda_j(A) \neq 0 \quad \forall i, j$$

Note: This is not fulfilled if the system has integrators!

System with Integrators

Consider a system on the following block triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad \begin{matrix} \lambda_i(A_{11}) \neq 0 & \forall i \\ \lambda_j(A_{22}) = 0 & \forall j \end{matrix}$$

where all integrators are collected in A_{22}. The corresponding Lyapunov equation for this system reads

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\mathsf{T} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\mathsf{T} & Q_{22} \end{bmatrix} \begin{bmatrix} A_{11}^\mathsf{T} & 0 \\ A_{12}^\mathsf{T} & A_{22}^\mathsf{T} \end{bmatrix} = - \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^\mathsf{T} & V_{22} \end{bmatrix}$$

which gives

$$A_{11}Q_{11} + Q_{11}A_{11}^\mathsf{T} = -V_{11} - A_{12}Q_{12}^\mathsf{T} - Q_{12}A_{12}^\mathsf{T}$$
$$A_{11}Q_{12} + Q_{12}A_{22}^\mathsf{T} = -V_{12} - A_{12}Q_{22}$$
$$A_{22}Q_{12}^\mathsf{T} + Q_{12}^\mathsf{T}A_{11}^\mathsf{T} = -V_{12}^\mathsf{T} - Q_{22}A_{12}^\mathsf{T}$$
$$A_{22}Q_{22} + Q_{22}A_{22}^\mathsf{T} = -V_{22}$$

Last equation has no unique solution since $\lambda_i(A_{22}) = 0 \quad \forall i$. But since A_{22} is nilpotent Q_{22} can be solved analytically!

Solution:

1. Find Q_{22} by using the analytical solution.
2. Find Q_{12} by solving a Sylvester equation.
3. Find Q_{11} by solving a Lyapunov equation.

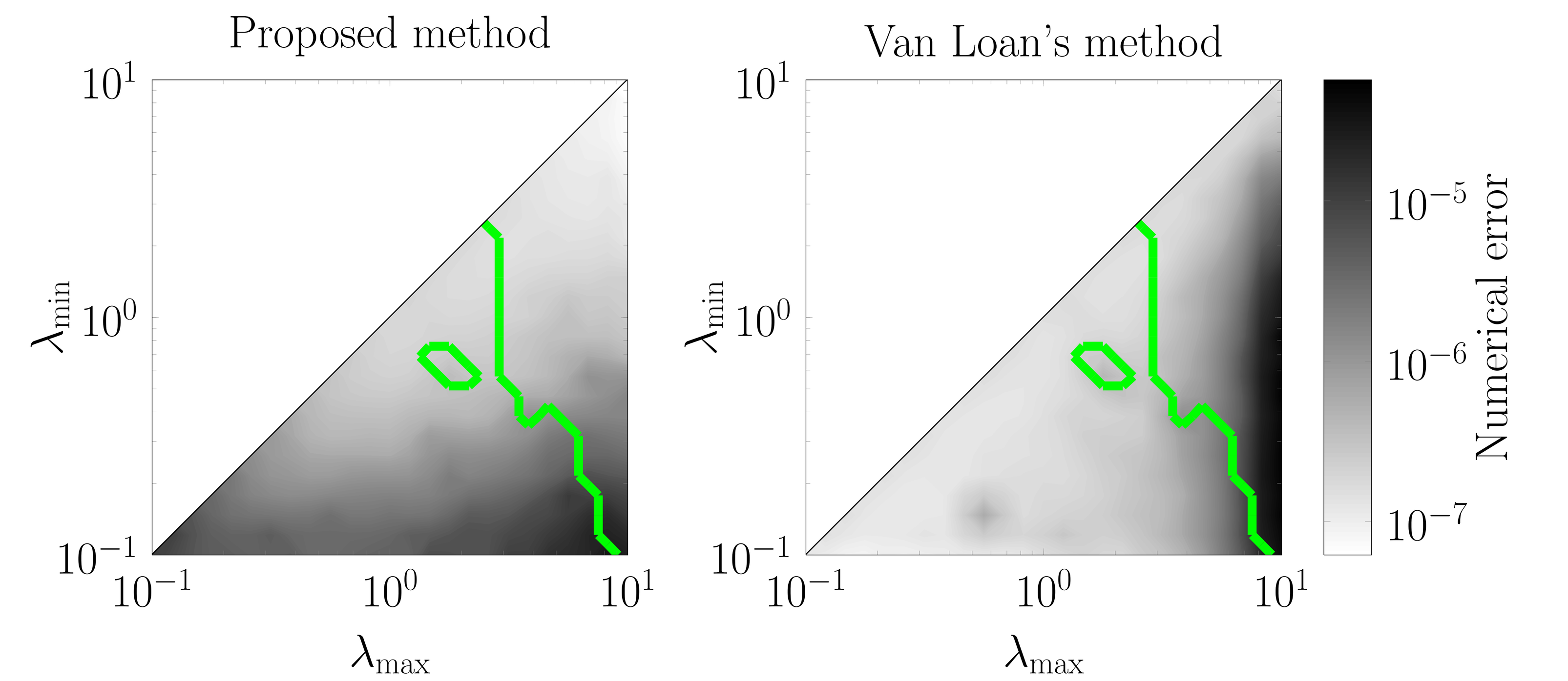
Van Loan's Method

The proposed method is compared with a standard method in the literature based on a matrix exponential of an augmented matrix

$$\begin{matrix} F_T = M_{11}, \\ Q_T = M_{12}M_{11}^\mathsf{T}, \end{matrix} \qquad e^{HT} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \qquad H = \begin{bmatrix} A & S \\ 0 & -A^\mathsf{T} \end{bmatrix}.$$

Numerical Evaluations

- Marginally stable systems with 4 stable poles and 2 integrators are considered.
- Each dimension in the 2D region $T\lambda_{\max}, T\lambda_{\min} \in [10^{-1}, 10^1]$ is divided into 25 bins, where
$$\lambda_{\max} = \max_i(|Re(\lambda_i)|) \quad \text{and} \quad \lambda_{\min} = \min_i(|Re(\lambda_i)|)$$
are the fastest and slowest stable pole, respectively.
- In total 100 systems are randomly generated for each bin.



According to the results, the proposed method performs better if the slowest pole is fast and/or the sampling is slow, whereas Van Loan's method performs better if the fastest pole is slow and/or the sampling is fast. Along the green line both methods perform equally well.

Conclusion

Numerical evaluations show that the proposed algorithm has advantageous numerical properties for slow sampling and fast dynamics in comparison with Van Loan's method.

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