Institutionen för Reglerteknik
Tekniska Högskolan i Lund

Linear Systems I

Exam March 6-17, 1995

Solutions to all problems should be well motivated. There is a total of 59 points, including 14 from the hand in problems. At least 30 should be reached for passed exam.

The examination time is 48 hours. Computers may be used and books may be consulted (except for the book where the appendix on four wheel car steering originates). You are encouraged to ask me if anything is questionable or difficult to understand, but you may not use help from each other.

I am grateful for your feedback on the course and would also be happy to have your errata collection for the book.

Good Luck!
Anders

1. Determine a minimal state space realization of the transfer matrix

$$\begin{bmatrix} \frac{2}{(s+3)(s+5)} & \frac{1}{s+3} & \frac{2(s+5)}{(s+1)(s+2)(s+3)} \\ \frac{2(s^2+7s+18)}{(s+1)(s+3)(s+5)} & \frac{-2s}{(s+1)(s+3)} & \frac{1}{s+3} \end{bmatrix}$$

and draw an illustrating block diagram.

(5 p)

2. Given $A \in R^{n\times n}$, $B \in R^{n\times m}$, with $\det(zI - A) \neq 0$ for $|z| \geq 1$, let

$$P = \sum_{k=0}^{\infty} A^k BB^T (A^k)^T$$

a. Prove that

$$P = APA^T + BB^T$$

b. Prove that there exists a $u(k)$ that drives the state $x(0) = 0$ of the system

$$x(k+1) = Ax(k) + Bu(k)$$

to $x(N) = \hat{x}$ for some $N < \infty$, if and only if $\hat{x}$ is in the range of P. (10 p)

3. For different values of ω, determine if the system

$$\ddot{x}(t) + [1 + \cos \omega t]\dot{x}(t) + x(t) = 0$$

is uniformly stable. (5 p)

4. Consider the possibly time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + e(t) \\ x(t)(0) &= x_0 \end{aligned}$$

for $t \in [0, T]$. The function $e(t)$, $t \in [0, T]$ describes measurement errors.

a. Derive the least-squares estimate of x_0 based on $y_{[0,T]}$ and $u_{[0,T]}$. The expression should be as explicit as possible.

b. Derive a weighting function $W(t, s)$, $t, s \in [0, T]$ such that the bound

$$\int_0^T \int_0^T e(t)^* W(t,s) e(s) ds dt \leq \epsilon \qquad (1)$$

is equivalent to

$$|x_0 - \hat{x}_0|^2 \leq \epsilon \qquad (2)$$

for the least squares estimate $\hat{x}_0$.

c. Determine the estimation maps and $W(t,s)$ for the example

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } C = B^* = [\,1 \quad 1\,]$$

d. Specify a "worst case" $e(t)$, that gives equality in both (**??**) and (**??**).

e. Repeat problem **c** for $A = 0$, $B = I$ and $C = [\,\sin t \quad \cos t\,]$ and the final time $T = 4\pi$. (15 p)

5. Consider the four wheel car steering model described in the appendix. Assume that the front steering angle δ_f is adjusted by an integrating motor with transfer function $1/s$. Thus, we have $\dot{\delta}_f = e_f$ and the plant model (A.2.7) with the performance variable a_f of (A.2.10) as output becomes

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \delta_f \end{bmatrix} + \begin{bmatrix} 0 & b_{12} \\ 0 & b_{22} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_f \\ \delta_r \end{bmatrix}$$

$$a_f = [\,c_1 \quad c_2 \quad d_1\,] \begin{bmatrix} \beta \\ r \\ \delta_f \end{bmatrix}$$

Find, under reasonable assumptions on the system coefficients, a feeback matrix K, defining the feedback law

$$\begin{bmatrix} e_f \\ \delta_r \end{bmatrix} = K \begin{bmatrix} \beta \\ r \\ \delta_f \end{bmatrix} + \begin{bmatrix} u_f \\ u_r \end{bmatrix}$$

such that

- r and δ_r become unobservable from a_f
- a_f becomes controllable from u_f but not from u_r

This type of feedback is used in four wheel steering buses, to let the driver control a_f via u_f, while a yaw control system uses u_r to take care of r. (10 p)