EXAM: CONVEX OPTIMIZATION FOR CONTROL

TIME: From 2004–06–16 until 2004–08–31 you may down-load the exam. After 36 hours you put the exam in an envelop, seal it, and hand it in. Exams should be handed in no later than 2004–08–31

EXAMINER: Anders Hansson, phone 013-281681, 070-3004401

RULES: Computers may be used and books may be consulted as well as all material handed out during the course. Any other printed material such as e.g. reports, journal articles or conference articles may not be consulted. You are encouraged to ask me if anything is questionable or difficult to understand, but you may not use help from each other or in any way discuss the exam with other people.

SOLUTIONS: Can be obtained by contacting the examiner in person. No solutions will be sent by E-mail.

RESULTS: Will be announced by E-mail.

PRELIMINARY GRADING RULES:	pass	$\geq$ 20 points
	fail	$\leq$ 19 points

NOTICE! Solutions have to be clearly motivated.

Good luck!

1. Do Exercise 7.3 in the book. (3p)

2. The vectors $x_0, \dots, x_k$ are said to be affinely dependent if $x_1 - x_0, \dots, x_k - x_0$ are linearly dependent. Show that the following conditions are equivalent.

   i $x_0, \dots, x_k$ are affinely dependent

   ii for some j it holds that $x_j = \sum_{i \neq j} \theta_i x_i$ and $\sum_{i \neq j} \theta_i = 1$

   iii
   $$\operatorname{rank}\left(\begin{bmatrix} x_0 & x_1 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right) < k+1$$

   (4p)

3. For $W = W^T \succ 0$ let $\mathcal{X} = \{x : x^T W x \leq 1\}$. Show that
   $$(A - BL)x \in \mathcal{X},\ \forall x \in \mathcal{X}$$
   if and only if
   $$(A - BL)^T W (A - BL) - W \preceq 0$$
   (4p)

4. The steepest descent method for minimizing the function $f(x)$ can be defined either using normalized or unnormalized directions. The normalized direction is given by
   $$v_{\mathrm{nsd}} = \operatorname{argmin} \left\{ \nabla f(x)^T v : \|v\|_2 \leq 1 \right\}$$
   and the unnormalized direction is given by
   $$v_{\mathrm{sd}} = \operatorname{argmin} \left(\nabla f(x)^T v + \frac{1}{2} \|v\|_2^2 \right)$$
   Show that $v_{\mathrm{sd}} = v_{\mathrm{nsd}} \|\nabla f(x)\|_2$. (4p)

5. Consider the LMI
   $$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0$$
   where $F_i = F_i^T$. Suppose that x does not satisfy the LMI. Show how to obtain a vector g that defines a cutting plane at x, i.e. find a vector g such that every feasible point for the LMI lies in $\{z : g^T(z - x) < 0\}$. (3p)

6. Consider the modified Lyapunov equation

$$PA + A^T P + I\,\mathbf{Tr}\,P + Q = 0$$

and derive an efficient way of solving this equation based on a solver for the ordinary Lyapunov equation

$$PA + A^T P + Q = 0$$

Your solution formula may only contain steps that involve solving ordinary Lyapunov equations and some additional trivial steps. These trivial steps must be computationally much cheaper than solving an ordinary Lyapunov equation. The cost of solving a Lyapunov equation is $\mathcal{O}(n^3)$, where n is the dimension of A. (4p)

7. (a) Show that the eigenvalues of a skew-symmetric matrix are all purely imaginary. (2p)

   (b) Assume that $Q \succeq 0$ and $R \succ 0$. Show that there exist $P = P^T$ such that

$$\begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & R \end{bmatrix} \succ 0$$

   if and only if (Q, A) has no unobservable modes corresponding to purely imaginary eigenvalues of A. (5p)

8. Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where $u(t) = \Delta(t)y(t)$ and $\|\Delta(t)\|_2 \leq 1$. A system is said to be quadratically stable if there is a quadratic Lyapunov function $V(x) = x^T P x$ that proves stability.

   (a) Show that quadratic stability of the linear system is equivalent to the existence of $P \succ 0$ such that

$$\begin{bmatrix} \xi \\ \pi \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \pi \end{bmatrix} < 0$$

for all nonzero ξ satisfying

$$\begin{bmatrix} \xi \\ \pi \end{bmatrix}^T \begin{bmatrix} -C^TC & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ \pi \end{bmatrix} \leq 0$$

(3p)

(b) Show that quadratic stability of the linear system is equivalent to the existence of $P \succ 0$ and $\tau \geq 0$ such that

$$\begin{bmatrix} A^TP + PA + \tau C^TC & PB \\ B^TP & -\tau I \end{bmatrix} \prec 0$$

(4p)

Hint: You may use the following version of the S-procedure: Assume that there exist ζ_0 such that $\zeta_0^T T_1 \zeta_0 > 0$. Then it holds that $\zeta^T T_0 \zeta > 0$ for all $\zeta \neq 0$ such that $\zeta^T T_1 \zeta \geq 0$ if and only if there exist $\tau \geq 0$ such that $T_0 - \tau T_1 \succ 0$.

(c) Show that quadratic stability of the linear system is equivalent to the existence of $P \succ 0$ such that

$$\begin{bmatrix} A^TP + PA + C^TC & PB \\ B^TP & -I \end{bmatrix} \prec 0$$

(2p)

(d) Show that quadratic stability of the linear system is equivalent to the existence of $P \succ 0$ such that

$$A^TP + PA + C^TC + PBB^TP \prec 0$$

(2p)

9. Consider the semidefinite program

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & F(x) \succeq 0 \\ & Ax = b \end{array}$$

where $F(x) = F_0 + \sum_{i=1}^n x_i F_i$, $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{p\times n}$, $b \in \mathbf{R}^p$, $c \in \mathbf{R}^n$, $F_i \in \mathbf{R}^{m\times m}$, $i = 1, 2, \ldots, n$. The dual program reads

$$\begin{array}{lll} \text{maximize} & -\,\mathbf{Tr} F_0 Z - b^T \mu & (1) \\ \text{subject to} & \mathbf{Tr} F_i Z - A_i^T \mu = c_i, \quad i = 1, 2, \ldots, n & (2) \\ & Z \succeq 0 & (3) \end{array}$$

where $Z = Z^T \in \mathbf{R}^{m\times m}$, $\mu \in \mathbf{R}^p$, and A_i is the ith column of A. Define the barrier transformation

$$\phi(x) = tc^T x - \log\det F(x)$$

for x such that $F(x) \succ 0$ and $t > 0$. Solutions of

$$\begin{array}{lll} \text{minimize} & \phi(x) & (4) \\ \text{subject to} & Ax = b & (5) \end{array}$$

for $t > 0$ define the central path of solutions. Let $\hat{x}$ be strictly feasible, i.e. $F(\hat{x}) \succ 0$ and $A\hat{x} = b$, and let v and ν be the Newton step and the associated dual variable of (4–5), respectively. They are solutions of

$$\begin{bmatrix} \nabla^2 \phi(\hat{x}) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ \nu \end{bmatrix} = - \begin{bmatrix} \nabla \phi(\hat{x}) \\ 0 \end{bmatrix}$$

(a) Show that

$$Z = \frac{1}{t}\left(X^{-1} - \sum_{j=1}^n X^{-1} F_j X^{-1} v_j \right)$$

$$\mu = \frac{1}{t}\nu$$

where $X = F(\hat{x})$, are feasible for the dual program (1–3) provided that the Newton decrement satisfies

$$\lambda(\hat{x}) = \sqrt{v^T \nabla^2 \phi(\hat{x}) v} < 1$$

(7p)

(b) Show that the duality gap

$$\eta = c^T \hat{x} + \mathbf{Tr} F_0 Z + b^T \mu$$

satisfies $0 \le \eta \le 2m/t$. (3p)