Storage Efficient Particle Filters for the Out of Sequence Measurement Problem

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***Abstract*—We propose a particle filter based solution which uses auxiliary fixed point smoothers to the problem of out of sequence measurements. Three different cases, namely, auxiliary extended Kalman smoother, auxiliary unscented Kalman smoother and auxiliary particle smoother are considered for the auxiliary fixed point smoother. The proposed filter which can effectively combine out of sequence measurements with arbitrary delay uses minimum storage requirements as opposed to a recently proposed alternative. The performance of our filters is compared to the extended version of the other alternative in the literature using a simulated scenario of a target with a highly nonlinear motion dynamics and measurement relations.**

Keywords: Out of sequence measurements, OOSM, particle filter, fixed point smoothing.

I. Introduction

Ability to process out of sequence measurements (OOSMs), which are the measurements of a target that arrive at a fusion center after a more current measurement of the same target has already been processed, is a requirement of modern multi-sensor platforms where discarding such measurements can cause significant decrease of performance. The common case of different data processing times and schemes of the sensors and/or generally unpredictable communication delays among the information sharing entities in distributed data processing environments brings up the problem of how to make efficient use of the information carried by this type of measurements.

There has been considerable amount of research done in this direction in the last decade thanks to the popularity of distributed sensing and tracking systems. The initial work on the subject concentrated on systems with linear state and measurement models for single lag OOSM problem. A survey of first suboptimal approaches is given in [1]. The surveyed solutions ranged from discarding the delayed observations to partial compensation of the effects of the process noise during the last lag. In fact, the difference between these methods are in the ways of taking care of the correlated process noise between the current time and the time (stamp) of the delayed measurement. An exact compensation of the process noise is given for the single lag problem in [2] with a comparison with the previous non-exact methods given in [3], [4]. It is concluded in [2] that the approach outlined in [1] as "Method 4" (given originally in [3]) is only slightly suboptimal and is "a reasonable compromise between simplicity and optimality" [2]. On the other hand the solution given in [4] is criticized as being excessively optimistic. The generalization of the approach of [2] to multiple lag problems is presented in [5]. Another solution based on state augmentation is discussed in [6], [7]. The general linear minimum mean square optimal solutions for the linear state and measurement equation case with multiple lags have been derived quite recently in [8] for different cases of stored (given) information. A one step solution to the multiple lag problem is proposed using the concept of "equivalent measurement" in [9], which reduced the storage and computation requirements of the algorithm of [5].

All of the above mentioned work had concentrated on the systems with linear state and measurement equations and it is quite straightforward to generalize them to the case of systems with linear state equations and nonlinear measurement equations using extended Kalman filter (EKF) or unscented Kalman filter (UKF) type approximations of the measurement equation. The common maneuvering target tracking methodology involving jump Markov linear systems [10], [11] has also been studied in the context of OOSM problem by some researchers (See [12] and the references therein). The case of general nonlinear discrete time systems has been first considered in connection with OOSMs in [13] where a particle filtering based solution, which modifies the final particle weights obtained before the arrival of the OOSM, is given. Later the method is equipped with a Markov chain Monte Carlo (MCMC) smoothing step in [14]. The particle filter used in [13], [14] needs excessive amount of storage resources by requiring all the previous particles to be stored for a predetermined maximum number of lags. An alternative particle filter has been proposed to circumvent this problem in [15] which has shown that the suggested particle filter has almost similar performance characteristics with an EKF based solution. Although the particle filters mentioned above do not require linear measurement equation and therefore do not suffer from approximation effects of EKF or UKF based measurement models, they implicitly require a linear state equation to operate. Specifically, the approach of [13], [14] requires a linear state dynamics to form the proposal density of the state corresponding to OOSM and the particle filter of [15] needs a state transition matrix to retrodict (predict backwards) the particles which is available only for systems with linear state dynamics. We give an extension of the proposal density generation of [13], [14] to nonlinear state dynamics in Sec. IV using EKF or UKF type approximations. However, even with this, a general nonlinear approach which is storage efficient for OOSM problem is still missing in the literature.

In this paper, we propose a particle filter based solution for OOSMs for the general nonlinear systems with less storage requirements than those of [13], [14]. The main idea of our study is based on the fact that storing the *measurements* for a predetermined maximum number of lags is much more storage efficient than keeping the particles and their weights for the same period in the particle filtering case. In fact, when there are infinite computation resources, the solution of the out of sequence measurement problem is trivial when the measurements (as well as particles) are stored since the reordering and reprocessing of the measurements would give the optimal minimum mean square estimate. We, here, instead, propose suboptimal approaches where sufficient statistics (like a single mean and covariance) are stored instead of previous particles and the computation requirements are also minimized by using auxiliary fixed lag smoothers which require much less computation than that would be required in the case reordering and reprocessing the measurements.

The outline of this document is given as follows. Section II makes a problem definition and illustrates the notation that will be used throughout the paper. In Section III, we give a brief summary of the approach of [13], [14] with an extension and then present our storage efficient particle filters. The proposed methods are compared on a simulated example in Section IV. The paper is finalized with conclusions in Section V.

II. Problem Formulation

We consider the following discrete-time nonlinear state space model defined on a probability space (Ω,$\mathcal{F}$,P)

$$x_{k+1} = f_{k+1,k}(x_k) + w_{k+1,k} \tag{1}$$

$$y_k = h_k(x_k) + v_k \tag{2}$$

where $\{x_k \in \mathbb{R}^n\}$ is the state sequence with initial distribution $x_0 \sim p_0(x_0)$. We here adopt an implicit simplified notation such that the system state dynamics given by (1) is a discretized version of a corresponding continuous time dynamics, i.e., $x_k \triangleq x(t_k)$ where $t_k \in \mathbb{R}$ is an arbitrary time value and $f_{k+1,k}(.)$ is the state transition function transforming $x(t_k)$ to $x(t_{k+1})$. We also assume that the time sequence $\{t_k\}_{k=0}^{\infty}$ is non-decreasing and therefore the transformation involved in (1) is not necessarily invertible. $\{y_k \triangleq y(t_k) \in \mathbb{R}^m\}$ is the noisy observation sequence,$\{w_{k+1,k} \in \mathbb{R}^n\}$ is a white process noise sequence with distribution $w_{k,k-1} \sim p^w_{k,k-1}(.)$. Here it is important to emphasize that $w_{k+1,k}$ models the lumped effects of a continuous independent increment process noise between the time instants t_k and t_{k+1}. $\{v_k \in \mathbb{R}^m\}$ is a white measurement noise sequence independent from the process noise with distribution $v_k \sim p^v_k(.)$. The functions $f_{k+1,k}(.)$ and $h_k(.)$ are measurable and in general nonlinear functions of the state x_k.

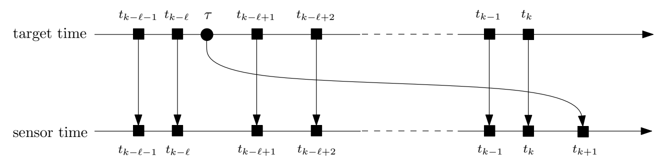

Figure 1. An illustration of ℓ-lag OOSM problem.

Suppose that we are at an intermediate stage of the estimation process in which we try to calculate (approximate) the posterior state density $p(x_m|y_{0:m})$ where $y_{0:m} \triangleq \{y_0, y_1, \ldots, y_m\}$. Assume that we have already processed k measurements $y_{0:k}$ and we estimated a posterior density $p(x_k|y_{0:k})$ of the state based on $y_{0:k}$. Then, suppose we get the $(k+1)$th measurement as an out of sequence measurement which we call as y_τ with a time stamp $\tau \in [t_{k-\ell}, t_{k-\ell+1})$. Now, the estimation algorithm is required to calculate (or approximate) the posterior density $p(x_k|y_{0:k,\tau})$ where $y_{0:k,\tau}$ is an abbreviation for $\{y_{0:k}, y_\tau\}$. The estimation problem posed by the scenario mentioned above is called the ℓ-lag OOSM problem and illustrated in Figure 1.

III. Particle Filters for OOSM

Particle filters [16]–[18] are Monte Carlo based statistical signal processing algorithms which provide a computationally attractive way of implementing the Bayesian density recursion given as

$$p(x_{0:k}|y_{0:k}) = \frac{p(y_k|x_{0:k}, y_{0:k-1})p(x_k|x_{0:k-1}, y_{0:k-1})}{p(y_k|y_{0:k-1})} \times p(x_{0:k-1}|y_{0:k-1}) \tag{3}$$

They do this by approximating the density functions $p(x_{0:k}|y_{0:k})$ as

$$p(x_{0:k}|y_{0:k}) \approx \sum_{i=1}^{N} w^{(i)}_{0:k}\delta(x_{0:k} - x^{(i)}_{0:k}) \tag{4}$$

where the samples $\{x^{(i)}_{0:k}\}_{i=1}^N$ are called particles and the scalars $\{w^{(i)}_{0:k}\}_{i=1}^N$ are the corresponding particle weights. Particle filters overcome the problem of sampling from the density function $p(x_{0:k}|y_{0:k})$ whose dimension increases at each time step by sampling from the so called proposal (importance) densities $\pi(x_{0:k}|y_{0:k})$ which satisfy

$$\pi(x_{0:k}|y_{0:k}) = \pi(x_k|x_{0:k-1}, y_{0:k})\pi(x_{0:k-1}|y_{0:k-1}) \tag{5}$$

At time k, this form of importance densities satisfy the equality

$$\int \pi(x_{0:k}|y_{0:k})dx_k = \pi(x_{0:k-1}|y_{0:k-1}) \tag{6}$$

and therefore make the resampling of the previous particles $\{x^{(i)}_{0:k-1}\}_{i=1}^N$ unnecessary since the samples $\{x^{(i)}_{0:k-1}\}_{i=1}^N$ are also distributed according to $\pi(x_{0:k}|y_{0:k})$ by (6). Therefore, at each time step k one has to sample only the last state components $\{x^{(i)}_k\}_{i=1}^N$ according to the density $\pi(x_k|x_{0:k-1}, y_{0:k})$.

The particle weights are updated as

$$w_{0:k}^{(i)} \propto \frac{p(y_k|x_{0:k}^{(i)}, y_{0:k-1})p(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{0:k-1})}{\pi(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{0:k})} w_{0:k-1}^{(i)} \quad (7)$$

with $\sum_{i=1}^{N} w_{0:k}^{(i)} = 1$.

Writing the recursion (3) for our case summarized in Section II and discarding unnecessary terms from the given conditions in each term, we get

$$p(x_{0:k,\tau}|y_{0:k,\tau}) = \frac{p(y_\tau|x_\tau)p(x_\tau|x_{k-\ell}, x_{k-\ell+1})}{p(y_\tau|y_{0:k})} p(x_{0:k}|y_{0:k}) \quad (8)$$

This is the main equation leading to the particle filter of [13], [14] for which we give a brief summary and make an extension in the following subsection.

A. Extension of the Approach of [13], [14]

The equation (8) suggests the sampling of $\{x_\tau^{(i)}\}$ from the density $p(x_\tau|x_{k-\ell}^{(i)}, x_{k-\ell+1}^{(i)})$ with a coefficient update given as

$$w_{0:k,\tau}^{(i)} \propto p(y_\tau|x_\tau^{(i)}) w_{0:k}^{(i)} \quad (9)$$

with $\sum_{i=1}^{N} w_{0:k,\tau}^{(i)} = 1$. The problem is now how to sample $x_\tau^{(i)}$ from the density function $p(x_\tau|x_{k-\ell}^{(i)}, x_{k-\ell+1}^{(i)})$. In [13], [14], this problem is investigated only for linear Gaussian system dynamics. In that case the density $p(x_\tau|x_{k-\ell}^{(i)}, x_{k-\ell+1}^{(i)})$ is a Gaussian density whose mean is a linear combination of $x_{k-\ell}^{(i)}$ and $x_{k-\ell+1}^{(i)}$ and covariance is based on the process noise covariances. This solution is based on the fact that

$$p(x_\tau|x_{k-\ell}^{(i)}, x_{k-\ell+1}^{(i)}) = \frac{p(x_{k-\ell+1}^{(i)}|x_\tau)p(x_\tau|x_{k-\ell}^{(i)})}{p(x_{k-\ell+1}^{(i)}|x_{k-\ell}^{(i)})} \quad (10)$$

Note that, instead of writing the Gaussian densities corresponding to $p(x_{k-\ell+1}^{(i)}|x_\tau)$, $p(x_\tau|x_{k-\ell}^{(i)})$ and then doing the multiplication, one can interpret (10) as the measurement updated version of the predicted density $p(x_\tau|x_{k-\ell}^{(i)})$ using the measurement

$$x_{k-\ell+1}^{(i)} = f_{k-\ell+1,\tau}(x_\tau) + w_{k-\ell+1,\tau}. \quad (11)$$

Therefore in the case of Gaussian densities $p^w(.)$, one can obtain approximate densities using EKF or UKF. For non-Gaussian case, one can approximate the involved densities using Gaussians and uses EKF or UKF, or make a simple particle approximation with a resampling step without the Gaussian approximation. We here illustrate the Gaussian case with EKF.

Calling the covariance of the noise terms w_{k_1,k_2} as Q_{k_1,k_2}, then the density $p(x_\tau|x_{k-\ell}^{(i)})$ is given exactly as

$$p(x_\tau|x_{k-\ell}^{(i)}) = \mathcal{N}(x_\tau; f_{\tau,k-\ell}(x_{k-\ell}^{(i)}), Q_{\tau,k-\ell}) \quad (12)$$

where the notation $\mathcal{N}(x;\bar{x},\Sigma)$ stands for a Gaussian probability density function which has a mean $\bar{x}$ and covariance Σ evaluated at the dummy variable x. Then updating this density using EKF formulas with measurement $x_{k-\ell+1}^{(i)}$, we get

$$p(x_\tau|x_{k-\ell}^{(i)}, x_{k-\ell+1}^{(i)}) = \mathcal{N}(x_\tau, \hat{x}_\tau^{(i)}, \Sigma_\tau^{(i)}) \quad (13)$$

where

$$\begin{aligned}\hat{x}_\tau =& f_{\tau,k-\ell}(x_{k-\ell}^{(i)}) \\ &+ K_\tau \left(x_{k-\ell+1}^{(i)} - f_{k-\ell+1,\tau}(f_{\tau,k-\ell}(x_{k-\ell}^{(i)}))\right)\end{aligned} \quad (14)$$

$$\begin{aligned}\Sigma_\tau =& Q_{\tau,k-\ell} \\ &- K_\tau(F_{k-\ell+1,\tau}Q_{\tau,k-\ell}F_{k-\ell+1,\tau}^T + Q_{k-\ell+1,\tau})K_\tau^T\end{aligned} \quad (15)$$

$$\begin{aligned}K_\tau =& Q_{\tau,k-\ell}F_{k-\ell+1,\tau}^T \\ &\times (F_{k-\ell+1,\tau}Q_{\tau,k-\ell}F_{k-\ell+1,\tau}^T + Q_{k-\ell+1,\tau})^{-1}\end{aligned} \quad (16)$$

$$F_{k-\ell+1,\tau} = \frac{\partial}{\partial x} f_{k-\ell+1,\tau}(x)\Big|_{x=f_{\tau,k-\ell}(x_{k-\ell}^{(i)})} \quad (17)$$

Due to the requirement of knowing $x_{k-\ell}^{(i)}$ and $x_{k-\ell+1}^{(i)}$, the particle filter of [13], [14] needs to store the particles for a predetermined maximum number of lags ℓ_{max} i.e., the storage requirements, denoted by Ω_{OM} using the initials of original authors, are $\Omega_{OM} = \{\{x_m^{(i)}\}_{i=1}^N\}_{m=k-\ell_{max}}^{k-1}$. This is an overwhelming storage requirement which makes one doubt about the gain obtained by including the OOSMs into the estimation process. In the next subsection, we present our approach of solving the same problem using a suboptimal method with sufficient statistics.

B. Storage Efficient Particle Filters

Since at each time instant, there is only one measurement y_k for the particle filter and since the dimension of measurements is usually less than that of states, storing the measurements instead of particles for a predetermined maximum number of lags is much more storage efficient *in the particle filtering case*. Therefore, we assume that the measurements $\{y_m\}_{m=k-\ell_{max}+1}^{k}$ are stored. Then our approach is based on the distribution of the posterior density $p(x_k|y_{0:k,\tau})$ using Bayes rule as follows

$$p(x_k|y_{0:k,\tau}) = \frac{p(y_\tau|x_k, y_{0:k})}{p(y_\tau|y_{0:k})} p(x_k|y_{0:k}) \quad (18)$$

which is actually a marginalized version of (8). Substituting the particle approximation of $p(x_k|y_{0:k})$ into (18), we obtain

$$p(x_k|y_{0:k,\tau}) = \sum_{i=1}^{N} \frac{p(y_\tau|x_k^{(i)}, y_{0:k})}{p(y_\tau|y_{0:k})} w_{0:k}^{(i)} \delta(x_k - x_k^{(i)}) \quad (19)$$

$$= \sum_{i=1}^{N} w_{0:k,\tau}^{(i)} \delta(x_k - x_k^{(i)}) \quad (20)$$

where

$$w_{0:k,\tau}^{(i)} \propto p(y_\tau|x_k^{(i)}, y_{0:k}) w_{0:k}^{(i)} \quad (21)$$

with $\sum_{i=1}^{N} w_{0:k,\tau}^{(i)} = 1$. Therefore, in order to update the particle approximation of $p(x_k|y_{0:k})$ to obtain the approximation of $p(x_k|y_{0:k,\tau})$ one has to calculate/approximate only the likelihoods $\{p(y_\tau|x_k^{(i)}, y_{0:k})\}_{i=1}^N$. In the following, we concentrate on how this calculation can be done using fixed point smoothers.

Using Bayes rule and the total probability theorem,

$$p(y_\tau|x_k^{(i)}, y_{0:k}) = \int p(y_\tau|x_\tau)p(x_\tau|x_k^{(i)}, y_{0:k})dx_\tau \quad (22)$$

$$= \int p(y_\tau|x_\tau)p(x_\tau|x_k^{(i)}, y_{0:k-1})dx_\tau \quad (23)$$

where y_k is dropped from the given condition since $x_k^{(i)}$ is given already. Hence, the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$ is dependent on a smoothed density of the state at the OOSM time stamp $p(x_\tau|x_k^{(i)}, y_{0:k-1})$. Here, both the measurements $y_{0:k-1}$ and the particle state $x_k^{(i)}$ must be interpreted as measurements i.e., the particle state $x_k^{(i)}$ is a measurement of the state x_{k-1} using the state dynamics $f(\cdot)$ as the measurement relation with

$$x_k^{(i)} = f_{k,k-1}(x_{k-1}) + w_{k,k-1}. \quad (24)$$

Since the density $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ is a smoothed density and since using the state value $x_k^{(i)}$ as a measurement must eliminate a possibly dominant multi-modal behavior of the density caused by processing only the measurements $y_{0:k-1}$, we, in the following, assume that it can be represented by a single Gaussian or with much less particles than the original particle filter has i.e., either

$$p(x_\tau|x_k^{(i)}, y_{0:k-1}) \approx \mathcal{N}(x_\tau; \hat{x}_{\tau|0:k-1,k^{(i)}}, \Sigma_{\tau|0:k-1,k^{(i)}}) \quad (25)$$

or

$$p(x_\tau|x_k^{(i)}, y_{0:k-1}) \approx \sum_{j=1}^{M} w_{\tau|0:k-1,k^{(i)}}^{(j)} \delta(x_\tau - x_{\tau|0:k-1,k^{(i)}}^{(j)}). \quad (26)$$

where $M \ll N$ is the number of particles in the auxiliary particle filter[1].

In the former case, the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$ is given with an EKF approximation of $p(y_\tau|x_\tau)$ as

$$p(y_\tau|x_k^{(i)}, y_{0:k}) = \mathcal{N}(y_\tau; \hat{y}_{\tau|0:k-1,k^{(i)}}, \Sigma_{\tau|0:k-1,k^{(i)}}^y) \quad (27)$$

where

$$\hat{y}_{\tau|0:k-1,k^{(i)}} = h_\tau(\hat{x}_{\tau|0:k-1,k^{(i)}}) \quad (28)$$

$$\Sigma_{\tau|0:k-1,k^{(i)}}^y = H_\tau \Sigma_{\tau|0:k-1,k^{(i)}} H_\tau^T + R_\tau \quad (29)$$

with $H_\tau = \frac{\partial h_\tau(x)}{\partial x}\Big|_{x=\hat{x}_{\tau|0:k-1,k^{(i)}}}$ and R_τ is the covariance of v_τ. UKF type approximation is also an alternative that can be applied in this situation.

[1] In this document, the term "auxiliary particle filter" is used in a completely different meaning than that of [19]. What we mean by it is "additional and/or supporting" particle filter.

In the latter case, i.e., in the case of particle approximation of $p(x_\tau|x_k^{(i)}, y_{0:k-1}))$ given by (26), we can calculate the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$ as

$$p(y_\tau|x_k^{(i)}, y_{0:k}) = \sum_{j=1}^{M} w_{\tau|0:k-1,k^{(i)}}^{(j)} p_\tau^v \left(y_\tau - h(x_{\tau|0:k-1,k^{(i)}}^{(j)})\right) \quad (30)$$

Since the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$ can be calculated by either (27) or (30) in the cases given by (25) and (26) respectively, we here turn to the problem of how to obtain the approximations (25) and (26) of $p(x_\tau|x_k^{(i)}, y_{0:k-1})$. We write $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ by partitioning $y_{0:k-1}$ as

$$p(x_\tau|x_k^{(i)}, y_{0:k-1}) = \frac{p(x_k^{(i)}, y_{k-\ell+1:k-1}|x_\tau)}{p(x_k^{(i)}, y_{k-\ell+1:k-1}|y_{0:k-\ell})} p(x_\tau|y_{0:k-\ell}) \quad (31)$$

This density update is a measurement update of the density $p(x_\tau|y_{0:k-\ell})$ using the measurements $\{x_k^{(i)}, y_{k-\ell+1:k-1}\}$. Therefore, we need a representation of the past i.e., $p(x_\tau|y_{0:k-\ell})$ to update with the measurements $\{x_k^{(i)}, y_{k-\ell+1:k-1}\}$. We here choose to summarize all the past information by a single mean $\hat{x}_{k|k}$ and covariance value $\Sigma_{k|k}$ for the state i.e., the stored information need to be

$$\Omega_{OG} = \left\{\{\hat{x}_{m|m}, \Sigma_{m|m}\}_{m=k-\ell_{max}}^k, \{y_m\}_{m=k-\ell_{max}+1}^k\right\}. \quad (32)$$

Updating the density $p(x_\tau|y_{0:k-\ell})$ to obtain $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ is a fixed point smoothing problem for which, we are going to consider three different cases, namely, extended Kalman smoother (EKS), unscented Kalman smoother (UKS), and particle smoother (PS). The fixed-point smoothing methods for the EKS and UKS approaches have been adapted from [20] as follows.

1) Auxiliary Extended Fixed-Point Kalman Smoother: The density $p(x_\tau|y_{0:k-\ell})$ is given using the EKF approximation as

$$p(x_\tau|y_{0:k-\ell}) = \mathcal{N}(x_\tau; \hat{x}_{\tau|k-\ell}, \Sigma_{\tau|k-\ell}) \quad (33)$$

where

$$\hat{x}_{\tau|k-\ell} = f_{\tau,k-\ell}(\hat{x}_{k-\ell|k-\ell}) \quad (34)$$

$$\Sigma_{\tau|k-\ell} = F_{\tau,k-\ell}\Sigma_{k-\ell|k-\ell}F_{\tau,k-\ell}^T + Q_{\tau,k-\ell} \quad (35)$$

with $F_{\tau,k-\ell} = \frac{\partial f_{\tau,k-\ell}(x)}{\partial x}\Big|_{x=\hat{x}_{k-\ell|k-\ell}}$. Then we define the augmented system state

$$\xi_k \triangleq \begin{bmatrix} \xi_k^1 \\ \xi_k^2 \end{bmatrix} = \begin{bmatrix} x_k \\ x_\tau \end{bmatrix} \quad (36)$$

with dynamics

$$\xi_{k+1} = \begin{bmatrix} \xi_{k+1}^1 \\ \xi_{k+1}^2 \end{bmatrix} = \begin{pmatrix} f_{k+1,k}(\xi_k^1) + w_{k+1,k} \\ \xi_k^2 \end{pmatrix} \quad (37)$$

with initial state and covariance

$$\xi_\tau = \begin{bmatrix} \hat{x}_{\tau|k-\ell} \\ \hat{x}_{\tau|k-\ell} \end{bmatrix}, \quad P_\tau = \begin{bmatrix} \Sigma_{\tau|k-\ell} & \Sigma_{\tau|k-\ell} \\ \Sigma_{\tau|k-\ell} & \Sigma_{\tau|k-\ell} \end{bmatrix} \quad (38)$$

The augmented system has the measurements

$$\eta_k^{(i)} = g_k^{(i)}(\xi_k^1) + \theta_k \tag{39}$$

where

$$g_m^{(i)}(x) = \begin{cases} h_m(x), & k-\ell+1 \le m \le k-2 \\ \begin{bmatrix} h_{k-1}(x) \\ f_{k,k-1}(x) \end{bmatrix}, & m = k-1 \end{cases} \tag{40}$$

$$\theta_m = \begin{cases} v_m, & k-\ell+1 \le m \le k-2 \\ \begin{bmatrix} v_{k-1} \\ w_{k,k-1} \end{bmatrix}, & m = k-1 \end{cases} \tag{41}$$

$$\eta_m^{(i)} = \begin{cases} y_m, & k-\ell+1 \le m \le k-2 \\ \begin{bmatrix} y_{k-1} \\ x_k^{(i)} \end{bmatrix}, & m = k-1 \end{cases} \tag{42}$$

Then, the smoothed density $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ is given as

$$p(x_\tau|x_k^{(i)}, y_{0:k-1}) = \mathcal{N}(x_\tau, \hat{\xi}_{k-1|k-1}^{(i),2}, P_{k-1|k-1}^{(i),22}) \tag{43}$$

where $\hat{\xi}_{k-1|k-1}^{(i),2}$ ($P_{k-1|k-1}^{(i),22}$) is the lower (bottom-right) block of the state estimate $\hat{\xi}_{k-1|k-1}^{(i)}$ (covariance $P_{k-1|k-1}^{(i)}$) obtained by processing the initial estimates ξ_τ and P_τ using EKF recursions on the augmented system with measurements $\{\eta_m^{(i)}\}_{m=k-\ell+1}^{k-1}$.

Remark 1: Note that the filters use the same initial estimates and the same measurements up to time $k-1$, i.e., the measurements $\{\eta_m^{(i)}\}_{m=k-\ell+1}^{k-2}$ are i-independent. Therefore a single filter is run until time $k-1$. Then the estimates $\hat{\xi}_{k-1|k-1}^{(i),2}$ and the covariances $P_{k-1|k-1}^{(i),22}$ are obtained using this common filter's estimate $\hat{\xi}_{k-1|k-2}$ and covariance $P_{k-1|k-2}$ by applying the EKF measurement update with the measurements $\{\eta_{k-1}^{(i)}\}_{i=1}^{N}$. □

Remark 2: In the case that $\tau \in [t_{k-1}, t_k)$, i.e., the OOSM belongs to the last sampling period, the filter has to use only $x_k^{(i)}$ as the measurement by updating the sufficient statistics of time $k-1$, with the measurement equation

$$x_k^{(i)} = f_{k,k-1}(x_{k-1}) + w_{k,k-1}. \tag{44}$$

□

In the context of (25), we have

$$\hat{x}_{\tau|0:k-1,k^{(i)}} = \hat{\xi}_{k-1|k-1}^{(i),2} \tag{45}$$

$$\Sigma_{\tau|0:k-1,k^{(i)}} = P_{k-1|k-1}^{(i),22} \tag{46}$$

and (27) can be used to obtain the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$.

2) Auxiliary Unscented Fixed-Point Kalman Smoother: The UKS case follows exactly the same lines as the EKS case except the initial estimates ξ_τ and P_τ are updated using UKF [21]–[23] recursions on the augmented system with measurements $\{\eta_m^{(i)}\}_{m=k-\ell+1}^{k}$.

3) Auxiliary Fixed-Point Particle Smoother: Note that, as mentioned in Section I, when the measurements are stored, one has the chance to order the measurements in the interval $[\tau, t_k]$ and redo all the particle filtering steps in between with ordered measurements. However, this really needs extensive computation facilities available. Here we propose an alternative solution with less computations. With the intuitive argument that the smoother density $p(x_\tau|x_k^i, y_{0:k-1})$ requires much less particles than the original particle filter, we start an auxiliary particle filter at time $k-\ell$ with number of particles set to $M \ll N$. The steps of the algorithm are given as follows.

Algorithm 1:
Some parts of the algorithm described below are over-summarized due to space considerations and details will be added in the final submission if space constraints allow us.

- Initialization:
  - Obtain M samples $\{\xi_{k-\ell}^{(j)}\}_{j=1}^M$ from the density $\mathcal{N}(x_{k-\ell}; \hat{x}_{k-\ell|k-\ell}, \Sigma_{k-\ell|k-\ell})$.
  - Predict the particles $\{\xi_{k-\ell}^{(j)}\}_{j=1}^M$ to time τ and set the weights of the particles according to the importance density used as usual. Note that these initial particles $\xi_\tau^{(j)}$ should always be kept track of until getting the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$.
- Recursion: For each m, $k-\ell+1 \le m \le k-1$
  - Set $r = \begin{cases} \tau, & t_{m-1} < \tau \\ m-1, & \text{otherwise} \end{cases}$
  - Predict the particles $\{\xi_r^{(j)}\}_{j=1}^M$ to time t_m set the weights of the particles according to the importance density used as usual.
  - Check resampling condition $\frac{1}{\sum_{j=1}^M (\mu_m^{(j)})^2} < M_{rs}$. If true, resample to obtain the new particles $\{\xi_m^{(j)}\}_{j=1}^M$. Keep track of the corresponding $\xi_\tau^{(j)}$ while choosing the new particles. Set $\mu_m^{(j)} = 1/M$ for $1 \le j \le M$. If false, set $\xi_m^{(j)} = \bar{\xi}_m^{(j)}$ for $1 \le j \le M$.
  - Increment m and go to the beginning of the recursion.
- Include $x_k^{(i)}$ as a measurement by updating the weights of the particles as usual by using the likelihood of $x_k^{(i)}$ derived from state dynamics (24).

Now, the approximated smoothed density $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ is given as

$$p(x_\tau|x_k^{(i)}, y_{0:k-1}) = \sum_{j=1}^{N} \mu_{k-1}^{(j,i)} \delta\left(x_\tau - \xi_\tau^{\left(\kappa(\xi_{k-1}^{(j)})\right)}\right) \tag{47}$$

where $\kappa(\xi_{k-1}^{(j)})$ is the index of the particle (in the set $\{\xi_\tau^{(j)}\}_{j=1}^M$) which corresponds to the updated particle $\xi_{k-1}^{(j)}$. In the context of (26), we have

$$w_{\tau|0:k-1,k^{(i)}}^{(j)} = \mu_{k-1}^{(j,i)} \tag{48}$$

$$x_{\tau|0:k-1,k^{(i)}} = \xi_\tau^{\left(\kappa(\xi_{k-1}^{(j)})\right)} \tag{49}$$

and (30) can be used to obtain the likelihood $p(y_\tau|x_k^{(i)}, y_{0:k})$.

Remark 3: Note that, until the inclusion of $x_k^{(i)}$ in the estimation process, the particles do not depend on i. Therefore, a single particle filter is run before the weights are updated based on $x_k^{(i)}$. As soon as weights are updated with $x_k^{(i)}$ the formula (30) can be used calculate the ith likelihood. It is also important to notice that the updated states of the particle filter do not carry any information other than the index of the corresponding initial particle.

IV. Simulation Study

In this section, the performances of the proposed methods are going to be compared on a simulated target tracking scenario. We consider a two-dimensional bearing only tracking problem with multiple sensors sending information to a common fusion center. The single target in the scenario makes a clockwise coordinated turn of radius $500m$ with a speed about $200km/h$ beginning in y-direction with the initial position $[-500m, 500m]$ for $30secs$. Two primary tracking sensors called as S_1 and S_2 acquire bearing data of the target corrupted by a Gaussian measurement noise with zero mean and standard deviation of $0.05rads$ with sampling period $T = 1sec$. The locations of the two sensors are selected to be $p_{S_1} = [-200m, 0m]$ and $p_{S_2} = [200m, 0m]$. The sensors are synchronized and their measurements are assumed to arrive at the fusion center without any delay. A third bearing sensor S_3 which is located at $p_{S_3} = [-750m, 750m]$ and have a standard deviation of $0.05rads$ is assumed to have serious communication problems with the fusion center. The behavior of this third sensor is modeled as follows. At each time value t_k that the fusion center gets information from the primary sensors about the target, an OOSM of the third sensor arrives at the fusion center with probability p_{oosm} and delay t_d. The time delay t_d of the OOSM is selected uniformly in the interval $(0, t_{max}]$ where t_{max} is a predefined maximum delay value. The fusion center then makes its filter updates first using the in sequence measurements coming from the primary sensors and then using the OOSM coming from the third sensor if any. The true target trajectory and the sensor positions used in the example are illustrated in Figure 2. The target motion is modeled in the filters with the discretized coordinated turn model with unknown constant turn rate (i.e., the turn rate is also a state variable) and with cartesian velocity. Therefore, the state of the target is given as $x_k = [p_k^x, p_k^y, v_k^x, v_k^y, \omega_k]^T$ where p,v and ω variables denote the position, velocity and turn rate respectively. In all simulations, we selected the standard deviations for the turn rate and speed as $\sigma_{\dot{\omega}} = 0.1rad/sec^2$ and $\sigma_{\dot{v}} = 10m/sec^2$ respectively.[2] The measurements are the standard bearing measurements with additive Gaussian noise with zero mean and standard deviation $0.05rads$ for all sensors. All the filters to be run are assumed to know nothing about the initial state of the target and therefore have been initialized with the state value $x_0 = [0, 0, 0, 0, 0]^T$ and a large covariance $P_0 = \text{diag}(250^2, 250^2, 30^2, 30^2, 0.1^2)$.

[2]Some of the state transition and covariance matrix descriptions are skipped for space considerations. The details will be added in the final submission if space constraints allow us.

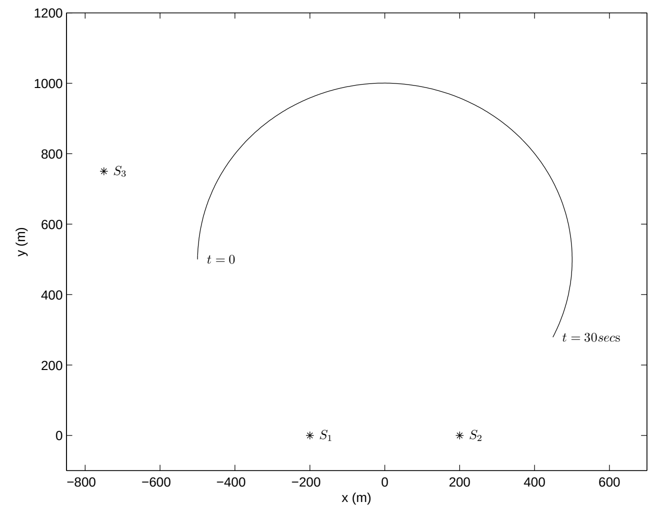

Figure 2. The target trajectory and the sensor positions used in the example.

It must be noted that for this state estimation problem with the model and the initial conditions described above, the EKF and UKF implementations failed to converge. The implementations of such filters also diverged with non-zero initial turn rate values, or even with the correct turn rate of the target. Note that there are several different UKF implementations in the literature. The ones implemented and observed to diverge in this study were the standard one given in [21], [22] and the modified version [23], with the standard parameters $\alpha = 10^{-3}$, $\beta = 2$ and $\kappa = 0$. As a result, only different particle filters have been run and compared in Monte Carlo simulations. Six different particle filters have been implemented:

- A SIR particle filter which discards the OOSMs and therefore processes only the measurements of the sensors S_1 and S_2. This filter is called as PFstd filter in the subsequent parts.
- An off-line SIR particle filter which collects all the measurements from the three sensors and then processes the time-ordered measurements. This filter is called as PFord filter in the subsequent parts.
- A SIR particle filter implementation of the extended version of the filter given in [13] proposed in Section III-A. This filter is called as PFoosmEXT filter in the subsequent parts.
- A SIR particle filter equipped with the auxiliary extended Kalman smoother described in Section III-B1 to process the OOSMs. This filter is called as PFoosmEKS filter in the subsequent parts.
- A SIR particle filter equipped with the auxiliary unscented Kalman smoother to process the OOSMs. This filter is called as PFoosmUKS filter in the subsequent parts.
- A SIR particle filter equipped with the auxiliary particle smoother described in Section III-B3 to process the OOSMs. This filter is called as PFoosmPS filter in the subsequent parts.

All the particle filters use 10000 particles. A very critical property for healthy operation of a particle filter is its sample diversity. During the extensive simulations, it was discovered that, at occasional times, an OOSM update can significantly reduce sample diversity which can cause significant performance deterioration. Although this was especially evident in PFoosmEXT filter, the other particle filters using the OOSMs have suffered from the same phenomenon which is caused either by the divergence of the OOSM update method or severe mismatch between the current particles and the OOSM, resulting in degenerate likelihood values, i.e., likelihood values (and hence updated weights) only few of which are non-zero. Inclusion of such OOSMs can reduce the estimation quality of the filters beyond even what can be achieved without using the OOSMs. Therefore, a detection mechanism for such individual cases has been designed to avoid OOSM updates which reduces the estimation quality. The detection process in our study works as follows:

- Before each OOSM update, calculate the number of effective samples using the non-updated weights $\{w_{0:k}^{(i)}\}_{i=1}^{N}$ as $N_{eff}^{prior} = 1/\sum_{i=1}^{N}(w_{0:k}^{(i)})^2$
- After the OOSM update, calculate again the number of effective particles using the updated weights $\{w_{0:k,\tau}^{(i)}\}_{i=1}^{N}$ as $N_{eff}^{posterior} = 1/\sum_{i=1}^{N}(w_{0:k,\tau}^{(i)})^2$
- Reject the OOSM if $N_{eff}^{posterior} < N_{eff}^{prior}/100$ and set $w_{0:k,\tau}^{(i)} = w_{0:k}^{(i)}$ for $i = 1, \ldots, N$.

The above procedure amounts to accepting only the OOSMs which do not decrease the number of effective samples by a hundred times or more.

The UKF used in the filter PFoosmUKS is the one proposed for solving the negative covariance matrices [23] with the standard parameter selections $\alpha = 10^{-3}$, $\beta = 2$ and $\kappa = 0$. The particle smoother used in the filter PFoosmPS is a standard SIR implementation with 1000 particles which is the one tenth of the number of particles used in the main filters.

A total number of 2000 Monte-Carlo runs have been made with the particle filters with changing the number OOSMs, OOSM arrival times and measurement noise realization affecting the sensor outputs in each run. The probability p_{oosm} of the arrival of OOSM from the sensor S_3 has been selected as 0.7. The maximum time lag t_{max} of the OOSMs is set to $5secs$. If an OOSM arrives from sensor S_3, its time stamp is selected to lie uniformly in the interval $[0, t_{max})$. With these selections, the smoothers must make at most five updates for each OOSM. The particle smoother used in PFoosmPS has the highest computation load which is limited to about half of a single update (both prediction and measurement updates) of the main particle filter.[3] Therefore, the computation load of our filters is also quite limited as well as the storage requirements.

[3]This is because the particle smoother has 1000 particles. For each OOSM, at most 5 prediction and measurement updates will be made, which makes a maximum number of 5000 prediction and measurements updates. Since the main particle filter has 10000 particles, it has to make 10000 prediction and measurement updates at a single step which is twice the maximum number of updates required for an OOSM.

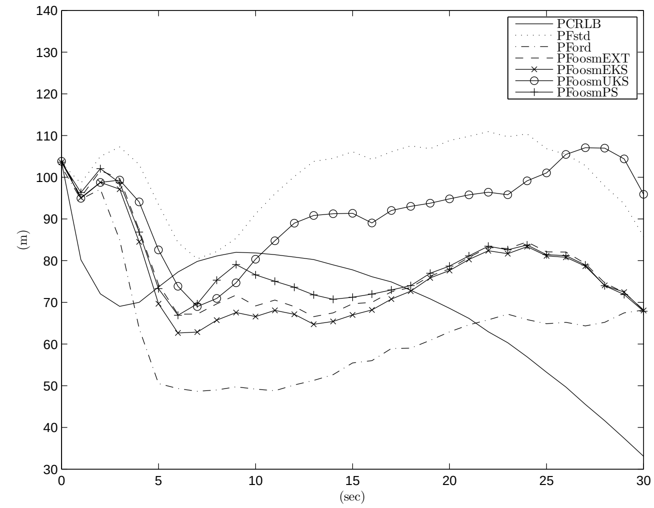

Figure 3. RMS position errors of the particle filters with PCRLB of the case without OOSMs.

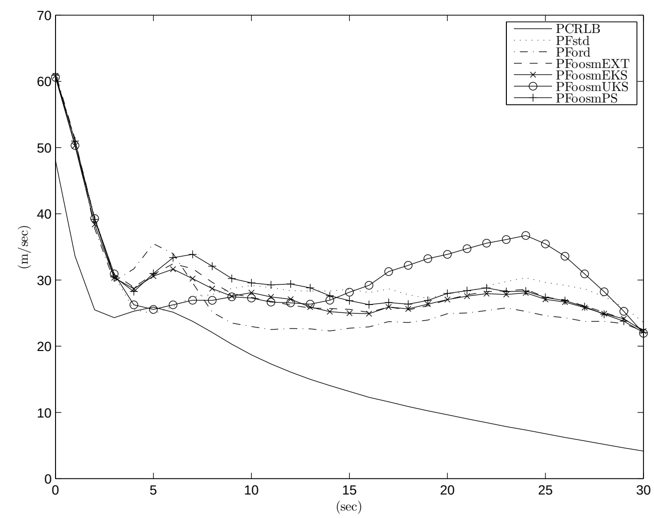

Figure 4. RMS velocity errors of the particle filters with PCRLB of the case without OOSMs.

The RMS position and velocity errors of the particle filters are illustrated in Figures 3 and 4 respectively. The parametric Cramer-Rao lower bounds (PCRLB) calculated for the case without OOSMs are also shown in the figures. As expected, the filter PFord which uses the ordered measurement data shows the best performance except for some transient behavior at the initial times of the velocity estimation. One must notice that in the case of velocity estimation, even the filter PFord with all its extra higher quality information cannot beat the PCRLB curve of case without OOSMs, which shows the degree of challenge involved in the problem. Moreover, there are isolated instances of lower performance of a filter using OOSMs (like PFoosmUKS in some periods of time in the position and especially velocity curves) than the filter PFstd which discards OOSMs. This clearly shows that OOSMs must be used carefully in the estimation process. The reasons about the failure of PFoosmUKS in incorporating OOSMs into the particles can be attributed partly to the extreme divergence of the UKF on the original target scenario and partly to the possible ineffectiveness of the OOSM discarding logic presented above in detecting this divergent cases. Requiring the OOSMs to keep the number of effective samples at a higher level by increasing the threshold in the logic can obviously reduce this phenomena. However, these cases are still worth illustration to point out the requirement of OOSM selection/discarding logics for similar applications using particle filters.

The EKS based filter PFoosmEKS obtains the best performance among the filters using OOSMs which actually confirms the fact that the smoothed density is $p(x_\tau|x_k^{(i)}, y_{0:k-1})$ uni-modal most of the times if not always. The slightly worse performance of the particle smoother based PFoosmPS than PFoosmEKS is quite expectable because Kalman filter based solutions of the smoothing problem might have better results than particle based solutions in the case when the smoothed density is uni-modal and especially if the number of particles is low. Except the filter PFoosmUKS, the estimation errors of our smoother based filters are quite comparable to those of PFoosmEXT which shows that they can serve as storage efficient alternatives to the existing methods.

V. Conclusions

This paper presents particle filters which use auxiliary fixed point smoothers for the OOSM problem. The methodology adopted allows the most general nonlinear state-space system framework with both system dynamics and measurement equations nonlinear. The resulting storage efficient particle filters are compared to an extended version of a previously proposed solution on a challenging simulated target tracking scenario. The results show that some of proposed solutions are at least capable alternatives to the existing ones with less storage requirements and moderate computational load. The results of the auxiliary extended Kalman smoother based particle filter is especially promising suggesting its use over the other algorithms.

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