FREQUENCY-DOMAIN CONTINUOUS-TIME AR MODELING USING NON-UNIFORMLY SAMPLED MEASUREMENTS

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ABSTRACT

A frequency domain approach to continuous-time auto regressive (AR) signal modeling is proposed. The algorithm allows for data pre-filtering as opposed to conventional AR modelling in the time domain. We illustrate the method by extracting resonance frequencies from data from a real-life application.

1. INTRODUCTION

A characteristic problem with signal processing using angle measurements on rotating axles is that the measurements are uniform in the angle domain but non-uniform (speed dependent) in the time domain. This comes from the fact that most common sensors for such applications measure the time between certain angle displacements, which is thus speed dependent. One can for instance illustrate this with the ABS sensors in a car, which give between 50 and 100 pulses per revolution for each wheel. Vibration analysis and other similar problems should be approached in the time sampled domain, so either one has to rely on data interpolation to uniform time sampling, or derive dedicated algorithms. Motivated by the recent advances in system identification in the frequency domain [1, 2], we present a frequency domain approach and compare it to a time domain algorithm proposed in [3, 4].

The main specifications on the algorithm aimed at high-sensitivity vibration analysis are as follows:

1. Being based on parametric physical models of the vibration.
2. Operate on short data batches in a pre-specified speed interval where the data pass several quality checks.
3. Reject wide band disturbances that are non-interfering with the vibration.
4. Reject narrow band disturbances that are interfering with the vibration.

The method given in [3, 4] successfully solves the first three specifications, but not the last one, while the proposed algorithm can easily be modified to eliminate out outliers in the frequency domain. This general problem occurs in several applications as in an automotive drive-line where the vibration indicates engine knocks, or in robotics where vibrations come from the load, just to mention a few. However, when coming to specific figures later on, our main focus is the same as in [4]. The vibration occurs in a pneumatic tire, which has several vibration modes. Further, there could be superimposed disturbances from engine, gear box, road and so on.

2. TIME AND FREQUENCY DOMAIN ALGORITHMS

2.1. Notation

Table 1 summarizes the notation and signal model that will be studied in the sequel. Basically, vibration analysis is approached by an autoregressive model motivated by a spring model of the axel and its contact paths with the surrounding. Superimposed on this signal are other vibrations and external disturbances, and the speed signal itself.

2.2. Algorithms

The time domain algorithm proposed in [3, 4] contains the following steps: (1) interpolate data to a a high sampling rate to avoid aliasing, (2) band-bass filter the signal to get rid of broad band disturbances and to focus on mode 2 in Table 3, (3) down-convert the signal utilizing deliberate aliasing, (4) estimate a discrete time AR model and (5) extract vibration data from this model. It is not easy to modify this algorithm to narrow-band interference, so the only practical solution is to turn off the algorithm when such a disturbance is detected.

Table 2 outlines the proposed frequency domain algorithm. This approach involves an interpolation step to approximate the Fourier transform. Since the number of sam-

Time domain	Frequency domain
$$y(t) = \frac{1}{A(p;\theta)}e(t) + d(t) \quad (1)$$ $$y[k] = y(t_k) = \frac{2\pi k}{L} + \int_{t_{k-1}}^{t_k} d(t)dt \quad (2)$$ $e(t)$ white noise, $A(p;\theta)$ AR model, θ parameters in the AR model, $d(t)$ disturbance, $y[k]$ measured non-uniform samples of angle, L number of cogs per revolution, Angle uniform sampling, not time uniform sampling.	$$\Phi_y(i\omega) = \Phi_H(i\omega)\Phi_e(i\omega) + \Phi_d(i\omega) \quad (3)$$ $$\Phi_H(i\omega) = \frac{1}{\lvert A(i\omega,\theta)\rvert^2} \quad (4)$$ $\Phi_e(i\omega) = 1$ white noise spectrum, $A(i\omega;\theta)$ AR model, θ parameters in the AR model, $\Phi_d(i\omega)$ disturbance spectrum, $\Phi_y(i\omega)$ 'measured' spectrum.

Table 1. Signal models and assumption in time and frequency domains, respectively.

ples is in the order of thousand per second for wheel speeds, the interpolation error and possible aliasing at the frequencies of interest is quite limited.

3. FREQUENCY DOMAIN ESTIMATION

Let us define Fourier transformation of the continuous time output $\{y(t) : t \in [0, T]\}$ in expression (1) in Table 1 above as

$$Y(i\omega) = \frac{1}{\sqrt{T}} \int_0^T y(t)e^{-i\omega t}dt.$$

A complicating element in a practical estimation procedure is that we do not have access to the entire continuous time realization of the output. Instead we have, as pointed out in expression (2) in Table 1, a finite number of samples of the continuous output y_t at time instances $\{t_1, t_2, \ldots, t_N\}$. Therefore it is in some way necessary to approximate or reconstruct the continuous time realization. In this paper the output is reconstructed as

$$\hat{y}(t) = \sum_{i=1}^{N} y(t_i)\phi_i(t - t_i)$$

where ϕ_i are interpolation kernels. We will use the piecewise-constant interpolation which will often go under the name Zero-Order Hold (ZOH) From the interpolated output it is possible to compute an approximation of the Fourier transform which is

$$\hat{Y}(i\omega) = \frac{1}{\sqrt{T}} \sum_{k=1}^{N-1} y(t_k) \frac{e^{-i\omega t_{k-1}} - e^{-i\omega t_k}}{i\omega}$$

in the piecewise constant case.

From the expressions above we can compute the approximate periodogram which we denote as

$$\hat{\hat{\Phi}}_y(i\omega) = \left|\hat{Y}(i\omega)\right|^2$$

The periodogram is an asymptotically unbiased estimate of the power spectrum.

In order to reduce the variance of the periodogram the data set is is split into N_b batches of duration T_R. Then a periodogram $\hat{\hat{\Phi}}_y^{(n)}$ is calculated for each batch and an estimate is formed as a direct average

$$\hat{\hat{\Phi}}_y(i\omega) = \frac{1}{N_b} \sum_{n=1}^{N_b} \hat{\hat{\Phi}}_y^{(n)}(i\omega).$$

This method is analogous to the method by Welch [5] for the smoothing of spectral estimates.

When an estimate of the power spectrum is available a continuous-time AR model can be identified by solving the following Maximum-Likelihood procedure described in [6]

$$\hat{\theta} = \arg\min_{\theta} \sum_{k \in \mathcal{N}} \frac{\hat{\hat{\Phi}}_y(i\omega_k)}{\Phi_H(i\omega_k, \theta)} + \log \Phi_H(i\omega_k, \theta).$$

Here Φ_H is defined as in (4) in Table 1. The frequencies $\omega_k,\ k = 1, \ldots, N_\omega$ where $\omega_k = 2\pi k/T,\ k \in \mathcal{N}$ have been selected such that the Fourier transforms of the output at different frequencies are asymptotically uncorrelated. The index set $\mathcal{K}$ denotes those frequencies we wish to use in the estimation procedure.

4. PROPERTIES OF BIAS AND VARIANCE

The bias or disturbances present in the periodogram will translate into bias in the parameter estimates. Therefore

1. Approximate continuous time Fourier transform with

$$\hat{Y}(i\omega_k) = \int_0^T \hat{y}(t)e^{i\omega_k t}dt, \tag{5}$$

$$\hat{y}(t) = \sum_{i=1}^{N} y(t_i)\phi_i(t-t_i), \tag{6}$$

$$w_k = \frac{2\pi}{T}k, \quad k \in \mathcal{N}. \tag{7}$$

2. Average the periodogram $\hat{\hat{\Phi}}_y(i\omega) = \left|\hat{Y}(i\omega)\right|^2$ over batches with similar vehicle speed.

3. Maximum likelihood estimate the CAR(2,v)-model

$$\hat{a} = \arg\max_a \sum_{k\in\mathcal{N}} \frac{\hat{\hat{\Phi}}_y(i\omega_k)}{\frac{\sigma^2}{|1+a_1(i\omega_k)+a_2(i\omega_k)^2|^2}} + \log\frac{\sigma^2}{|1+a_1(i\omega_k)+a_2(i\omega_k)^2|^2} \tag{8}$$

Table 2. Frequency-Domain algorithm.

it would be useful to a user of the method to know how these relate to each other in order to minimize the bias. One would also like to tune the estimation procedure in order to minimize the variance of the parameter estimates. The results below can be found in the report by [6] where in the case of bias it is possible to show that

$$E(\hat{\theta}-\theta^*) \approx \sum_{k\in\mathcal{N}} \Psi(i\omega_k)\Delta\Phi_y(i\omega_k).$$

where $\hat{\theta}$ are the estimated parameters and θ^* are the true parameter values. The relative bias in the periodogram estimate of the power spectrum is defined as

$$\Delta\Phi(i\omega_k) = \frac{E\hat{\hat{\Phi}}_y(i\omega_k) - \Phi(i\omega_k,\theta^*)}{\Phi(i\omega_k,\theta^*)}$$

Here we have for simplicity defined $\Phi = \Phi_H$. The sensitivity of the parameter estimates to the relative bias in the periodogram is

$$\Psi(i\omega_k) = M(\theta^*,\Phi)^{-1}M_k(i\omega_k,\theta^*,\Phi)$$

Here the so called relative sensitivity is

$$M_k(i\omega_k,\theta^*,\Phi) = \frac{\Phi'_\theta(i\omega_k,\theta^*)}{\Phi(i\omega_k,\theta^*)}.$$

and

$$M(\theta^*,\Phi) = \sum_{k\in\mathcal{N}} M_k(i\omega_k,\theta^*,\Phi)M_k(i\omega_k,\theta^*,\Phi)^T$$

Hence, in order to avoid bias due to disturbances etc. it is necessary to ignore information from frequencies where the relative bias and sensitivity are large.

In the case of variance we resort to asymptotics where it is possible to show that

$$E(\hat{\theta}-\theta^*)(\hat{\theta}-\theta^*)^T \to M(\theta^*,\Phi)^{-1}.$$

Again the relative sensitivity plays an important role. In order to reduce the variance information from frequencies where the relative sensitivity is large should be included. This together with the rules of thumb on bias necessitates a bias variance tradeoff to be made.

5. EXPERIMENTAL RESULTS

In this section we apply the theory developed above to the estimation of the resonance peak of the vertical vibrations of a pneumatic tire. The samples $y(t_k)$ are pre-processed measurements from an axel angle measurement device. The frequency spectrum of $y(t)$ is approximately divided as summarized in Table 3.

0-10	10-15	15-30	30-60	60-80	80-100	100–
Speed	Mode 1	Noise	Mode 2	Noise	Mode 3	Noise
Narrow-band noise components						

Table 3. Frequency spectrum with approximate limits in Hz

The vibrations in the range 30-60 Hertz can be modelled as a spring-damper system excited by white noise e(t)

$$y(t) = H(p)e(t)$$

with transfer operator

$$H(p) = \frac{\sigma}{p^2 + 2\gamma p + \omega_0^2}.$$

The output will therefore have the continuous-time spectrum

$$\Phi_H(i\omega) = \frac{\sigma^2}{(\omega^2-\omega_0^2)^2 + 4\gamma^2\omega^2}. \tag{9}$$

with a resonance peak located at the frequency

$$\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2}$$

For the special parameterization of the spectrum in expression (9) the relative sensitivity functions with respect to respective parameters are shown in Figure 1. Here we have chosen $\gamma = 33.88$ and $\omega_0 = 289.687$. This means that $w_{res} = 285\ rad/s$ or $f_{res} = 45.47Hz$. From this figure, we conclude that γ is sensitive near the natural resonance frequency of the system. The frequency ω_0 on the other

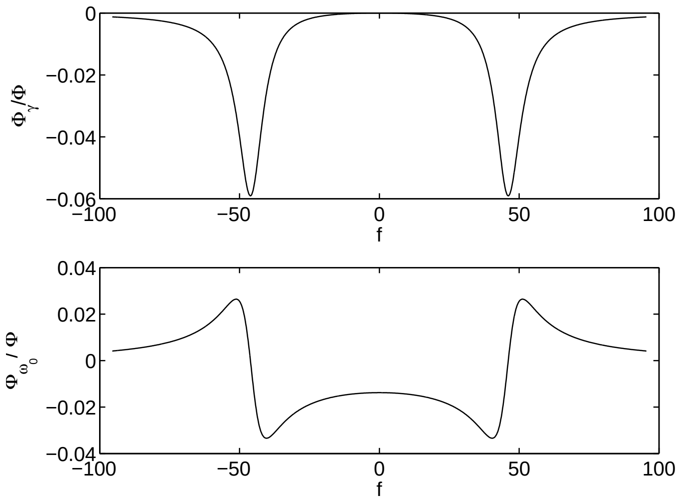

Fig. 1. Relative sensitivities for γ (upper) and ω_0 (lower).

hand is particularly sensitive at low frequencies. According to Table 3 there is noise between 15 and 30 Hertz and 60 and 80 Hertz. Therefore we restrict the frequencies used to those between 30 and 60 Hertz.

In Figure 2 we have estimated the resonance frequencies from the refined set of real life data from an ABS sensor. The data have been divided into four parts. These parts have then been subdivided into a set of batches with a duration of a certain number of revolutions or laps of the tire. The number of laps per sub-batch is indicated on the x-axis of the figure. Periodograms have been estimated using ZOH for each batch and subsequently averaged in order to yield four estimates of the spectrum. The result indicates that the

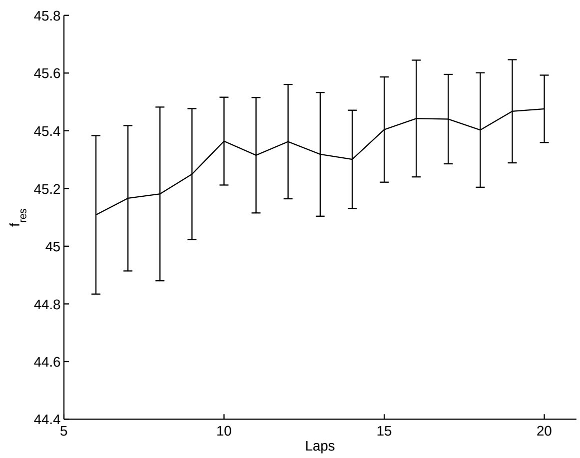

Fig. 2. Resonance frequency f_{res} in Hertz versus batch size in number of tire laps. Bars indicates one standard deviation.

method is feasible and that a batch size of about 10 laps is sufficient to yield a stable estimate of the resonance frequency with moderate variance. The bias encountered in the left part of the plot is due to the small number of frequency domain samples combined with the well known $\sqrt{N}$ consistency of the Maximum Likelihood method used.

6. CONCLUSIONS

In this paper we have outlined a frequency-domain alternative to time-domain estimation of axel vibrations. The method can easily be extended in order to reject narrow band disturbances interfering with the vibration. The method has proved feasible on real-life data. An algorithm which is more robust to outliers can be aquired if the probability density function in the ML-method is changed to a more appropriate one [7]. This would be a natural objective of future work.

7. REFERENCES

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