# POSITIONING USING TIME-DIFFERENCE OF ARRIVAL MEASUREMENTS

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## ABSTRACT

The problem of position estimation from Time Difference Of Arrival (TDOA) measurements occurs in a range of applications from wireless communication networks to electronic warfare positioning. Correlation analysis of the transmitted signal to two receivers gives rise to one hyperbolic function. With more than two receivers, we can compute more hyperbolic functions, which ideally intersect in one unique point. With TDOA measurement uncertainty, we face a non-linear estimation problem. We here suggest and compare both a Monte Carlo based method for positioning and a gradient search algorithm using a non-linear least squares framework. The former has the feature to be easily extended to a dynamic framework where a motion model of the transmitter is included. A small simulation study is presented.

## 1. INTRODUCTION

Figure 1 illustrates how two cooperating receivers can calculate a path difference from the time-difference of arrival, and how this path difference corresponds to a hyperbolic function [1]. We point out two important applications where this problem occurs:

- Mobile terminal positioning. The available measurements are either network-assisted or mobile-assisted, using up-link or down-link information. For an overview, see [2, 3]. The current 'yellow page' services are based on *Received Signal Strength* (RSS) of signals with known powers and a course *Angle* measurement from the sector antenna [4, 5]. This gives a quite course estimate. It is well-known [6, 4, 1] that a much higher accuracy rather insensitive to fading is based on time of arrival (TOA). In future system, only the time difference of arrival (TDOA) or enhanced observed time difference (E-OTD) measurements may be possible to compute. Another source of information for tracking moving objects is map information, see [7].

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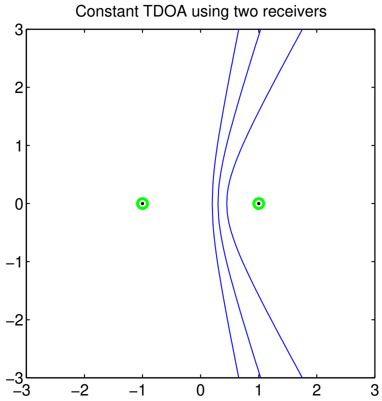



**Fig. 1**. The hyperbolic function representing constant TDOA for three different TDOA's (0.4, 0.6 and 0.9 scale units, respectively).

- Electronic warfare, where the problem is to accurately locate enemy transmitters to be able to make appropriate countermeasures. The TDOA approach may here offer higher accuracy than traditional triangulation approaches, where the angular measurements potentially have larger inaccuracy than TDOA measurements.

The TDOA measurement is computed as follows:

1. The sender transmits a signal $s(t)$ which is delayed $\tau_i$ to receiver $i$ according to distance to each receiver. The signal can either be a pilot from a mobile (uplink), where the mobile's absolute time is unknown, or it can be unknown, as is the case in electronic warfare. In either case, $\tau_i$ cannot be computed.

2. Correlation analysis provides a time delay $\tau_i - \tau_j$ corresponding to the path difference to receivers $i$ and $j$.

In [8], a general framework covering all these kind of measurements (TOA, TDOA, E-TDOA, RSS, Angle, Map) is given, *provided* that a motion model for the transmitter is available. That is, a dynamic framework is assumed, and the particle filter is suggested. We here study the static case, where we have only one TDOA measurement for each pair of receivers. We will compare a particle filter based estimator with a least mean square (LMS) algorithm in a non-linear least squares framework.

## 2. TDOA MEASUREMENTS

The received signals are

$$y_i(t) = a_i s(t - \tau_i) + e_i(t), \quad i = 1, 2, \ldots n \qquad (1)$$

where the receiver $i$ is located at $x_i, y_i$ and the transmitter is in $x, y$, which is unknown.

With known reference $s(t)$ and perfect synchronization, we can directly estimate $\tau_i$ (TOA), and estimate $(x, y)$ using a non-linear least squares framework, similar to GPS.

With unknown reference, the simplest idea is to compare the received signals pairwise. Assume a correlation function that pairwise computes an estimate of

$$\Delta d_{i,j} = v(\tau_i - \tau_j), \quad 1 \le i < j \le n. \qquad (2)$$

where $v$ is the speed of sound, light or water vibrations. Here, $n$ is the number of receivers and $(i, j)$ is an enumeration of all $K$ pair of receivers, where

$$K = \binom{n}{2} \qquad (3)$$

Each $\Delta d_{i,j}$ corresponds to positions $(x, y)$ along a hyperbola.

Assume first that the receivers are both located at the $x$-axis at $x = D/2$ and $x = -D/2$, respectively. The hyperbolic function can then be expressed as

$$\begin{aligned} d_2 &= \sqrt{y^2 + (x + D/2)^2}, && (4a)\\ d_1 &= -\sqrt{y^2 + (x - D/2)^2}, && (4b)\\ \Delta d &= d_2 - d_1 = h(x, y, D) && (4c)\\ &= \sqrt{y^2 + (x + D/2)^2} - \sqrt{y^2 + (x - D)/2)^2}. && (4d) \end{aligned}$$

After some simplifications, this equation can be rewritten in a more compact form as

$$\frac{x^2}{a} - \frac{y^2}{b} = \frac{x^2}{\Delta d^2/4} - \frac{y^2}{D^2/4 - \Delta d^2/4} = 1. \qquad (5)$$

The solution to this equation has asymptotes along the lines

$$y = \pm \frac{b}{a} x = \pm \sqrt{\frac{D^2/4 - \Delta d^2/4}{\Delta d^2/4}} x. \qquad (6)$$

which defines the angle of arrival for far-away transmitters. Figure 1 illustrates the hyperbolic function in the local coordinate system $(x, y)$.

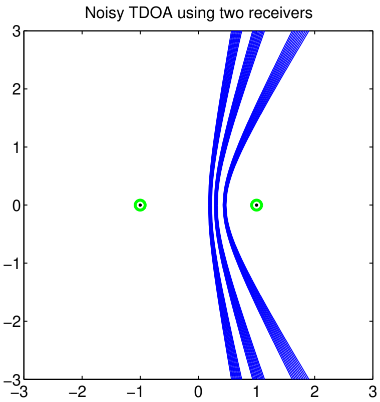



**Fig. 2**. Same as Figure 1, but with measurement uncertainty in $\Delta d$.

For a general receiver position, we simply translate the hyperbolic function (5) in local coordinates $(x, y)$ to global coordinates $(X, Y)$ using

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (7)$$

where $X_0 = (X_i + X_j)/2$, $Y_0 = (Y_i + Y_j)/2$ locates the center point of the receiver pair. The hyperbolic function in global coordinates is thus given by $\Delta d_{i,j} = h(x, y, D)$ in (5), with

$$\begin{aligned} D &= \sqrt{(Y_i - Y_j)^2 + (X_i - X_j)^2} && (8a)\\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} X - X_0 \\ Y - Y_0 \end{pmatrix} && (8b)\\ \alpha &= \arctan\left(\frac{Y_i - Y_j}{X_i - X_j}\right) && (8c) \end{aligned}$$

We have now a functional form suitable for representing the measurement uncertainty in TDOA, which implies an uncertain hyperbolic area rather than a line. This is illustrated in Figure 2. The farther away along the asymptotes, the larger absolute uncertainty in position.

## 3. THE NON-LINEAR LEAST SQUARES PROBLEM

The general problem is to solve the (possibly over-determined) non-linear system of $K$ equations

$$\Delta d_{i,j} = h(X, Y; X_i, Y_i, X_j, Y_j), \quad 1 \le i < j \le n \qquad (9)$$

for the sender position $(X, Y)$, given the receiver positions $(X_i, Y_i)$. Now, the non-linear least squares estimate of $(X, Y)$ is given by

$$(\hat{X}, \hat{Y}) = \arg \min_{(X,Y)} \sum_{i>j} (\Delta d_{i,j} - h(X, Y; X_i, Y_i, X_j, Y_j))^2$$

To simplify notation, we use $P = (X, Y)$ for the position. Then, rewrite the minimization problem in vector notation using a *weighted* least sqauares criterion

$$\hat{P} = \quad = \arg\min_{P} (\Delta d - h(P))^T R^{-1} (\Delta d - h(P)) \quad (10)$$

where $\Delta d = (\Delta d_{1,2}, \ldots \Delta d_{n-1,n})^T$ and $R = \mathrm{Cov}(\Delta d)$ is the covariance matrix for the TDOA measurements. The solution defines the minimum variance estimate. For a Gaussian assumption on the TDOA noise, this coincides with the maximum likelihood estimate.

Using the assumption that $\Delta d = h(P_0) + e$, where $P_0$ is the true position and the TDOA noise has $\mathrm{Cov}(e) = R$, a first order Taylor expansion around the true value gives $h(P) \approx h(P_0) + h'_P(P_0)(P - P_0)$. The least squares theory now gives

$$\mathrm{Cov}(\hat{P}) = (h'_P(P_0))^{\dagger} R ((h'_P(P_0))^{\dagger})^T, \qquad (11)$$

provided that $\hat{P}$ is sufficiently close to the true position (high SNR). For Gaussian noise $e$, this expression also defines the Cramér-Rao lower bound. That is, no estimator can perform better than this bound, given that we have find a small enough neighborhood of the true position. From (11), we can obtain guidelines for what a favourable $P_0$ is, or for electronic warfare how to place the receivers in the best possible way.

## 4. ALGORITHMS

Three different approaches have been compared:

1. Compute the intersection point of each pair of hyperbolic functions. There are

$$\begin{pmatrix} K \\ 2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} n \\ 2 \end{pmatrix} \\ 2 \end{pmatrix} \qquad (12)$$

   pair of hyperbolic functions. The position estimate can then be the (weighted) average of these points. Since each pair can have no, one or two intersections, the logic to find the correct one is non-trivial. Further, it is a bit complicated to find the correct weighting.

2. Applying the stochastic gradient algorithm to the non-linear least squares problem.

3. Numerical approximation of the non-linear least squares problem using Monte Carlo based techniques. This is refered to as the particle filter (PF), which is a static version of the general approach described in [8].

The first approach has shown to give inferior results than the other two. The latter two ones are described below.

The normalized gradient algorithm can be written as follows:

**Algorithm 1 (Stochastic gradient algorithm)**

$$P^{(m+1)} = P^{(m)} - \mu^{(m)} h'_P(P^{(m)})(\Delta d - h(P^{(m)})) \quad (13)$$

*A good step size can be computed by bisection techniques, line search or, as in the simulations, as the normalized LMS step size*

$$\mu^{(m)} = \frac{\mu}{(h'_P(P^{(m)}))^T h'_P(P^{(m)})} \qquad (14)$$

The particle filter is a static version of the well-known SIR algorithm [9, 10].

**Algorithm 2 (Static particle filter)** .

1. *Randomize $N$ 'particles' (here possible positions) $P^i$.*

2. *Choose jittering constants $C_R$ and $C_Q$ and let the position random walk covariance $\bar{Q} = C_Q/k^2$ and jittering measurement noise $\bar{R} = R + C_R/k^2$. The idea with jittering noise is to explore a smaller and smaller neighborhood more and more accurately.*

3. *Iterate for $k = 1, 2, \ldots$ until $\hat{P}(k)$ has converged.*

   (a) *Compute the particle weights $w^i$ using the likelihood*

   $$w^i = \exp\left((\Delta d - h(P^i))^T R^{-1} (\Delta d - h(P^i))\right)$$

   *and normalize $w^i = w^i / (\sum w^i)$.*

   (b) *Compute the estimate $\hat{P}(k) = \sum_i w^i P^i$.*

   (c) *Resample with replacement the particles, where the probability to pick one particle is proportional to its weight. After the resampling, the weigths are reset $w^i = 1/N$.*

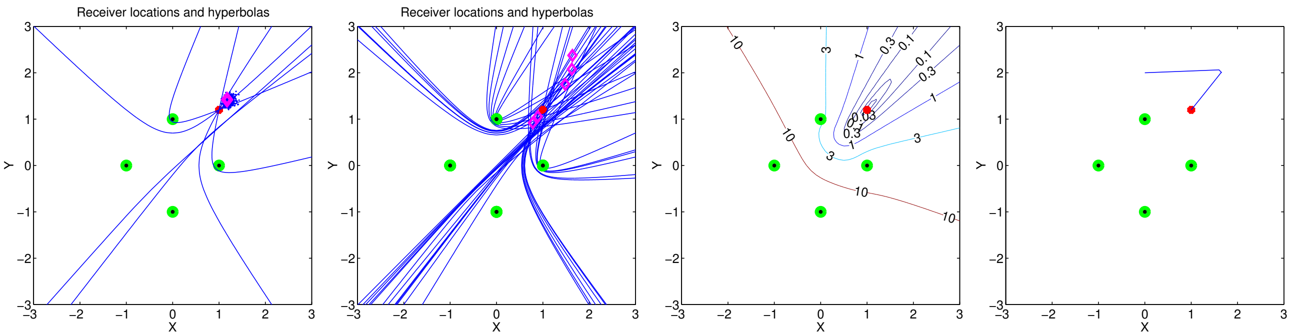
   (d) *Spread out the particles as $P^i = P^i + w$, where $w \in \mathrm{N}(0, \bar{Q})$.*

The resampling step is the key to get a working algorithm. In the standard particle filter, $k$ denotes time and there is a time update step where the particles are moved according to a velocity measurement and a movement noise $w$, otherwise the algorithms are quite similar. Compare to the particle filters in [7, 8].

## 5. SIMULATIONS

Figure 3(a) illustrates the test scenario, with four receivers computing in total six TDOA measurements.

Figure 3(b) shows what happens to the hyperbolic functions when Gaussian measurement noise (standard deviation of 0.1 scale units) is added to the TDOA measurements. That is, there is no clear cut intersection point. To understand the non-linear least squares criterion, the level curves




(a) (b) (c) (d)

**Fig. 3**. (a) Test scenario: four receivers are placed in a square, and six resulting hyperbolic functions from noise-free TDOA:s intersect at the transmitter position. Also shown is the particle cloud and the resulting position estimate using Algorithm 1. (b) Same as (a), but the hyperbolic functions are computed from six different noisy TDOA vectors. This illustrates that in general there is no unique intersection of all six lines. (c) Contour plot of non-linear least squares criterion $\sum_{i<j}(\Delta d_{i,j} - h(X,Y))^2$. In this sceario, there is no local minima and a gradient algorithm will converge from any initialization. (d) Gradient search using a normalized least mean square method on Algorithm 2 (compare the path to the contour plot in (c)).

of (10) are plotted in Figure 2(c). This plot explains how the weights of the particles are computed, and in which direction the gradient in LMS points.

Figure 3(a) shows the particle cloud after a few iterations. As might be expected from the level curves, this cloud is often found beyond the true position, as in this example. However, in most cases the true position is found. Figure 3(d), finally, shows the learning path of LMS. For this scenario which lacks local minima, LMS is to prefer.

## 6. CONCLUSIONS

Two algorithms have been suggested for finding the position of a transmitter, given TDOA measurements computed from the received signal for at least three receivers. A non-linear least squares approach was advocated, enabling local analysis yielding a position covariance and a Cramér-Rao lower bound. A simulation study illustrated the TDOA problem in general and the performance of the two suggested algorithms. On-going work aims at investigating the practical performance for acoustic communication, how to use the Cramér-Rao bound to determine receiver locations and use Monte-Carlo simulations to examine the performance of the two algorithms.

## 7. REFERENCES

[1] M.A. Spirito and A.G. Mattioli, "On the hyperbolic positioning of GSM mobile stations," in *Proc. International Symposium on Signals, Systems and Electronics*, Sept 1998.

[2] C. Drane, M. Macnaughtan, and C. Scott, "Positioning GSM telephones," *IEEE Communications Magazine*, vol. 36, no. 4, 1998.

[3] M. Silventoinen and T. Rantalainen, "Mobile station locating in GSM," in *Proc. IEEE Wireless Communication System Symposium*, Nov 1995.

[4] M.A. Spirito, "Further results on GSM mobile station location," *IEE Electronics Letters*, vol. 35, no. 22, 1999.

[5] B. Mark and Z. Zaidi, "Robust mobility tracking for cellular networks," in *Proc. IEEE International Communications Conference*, New York, NY, USA, 2002.

[6] S. Fischer, H. Koorapaty, E. Larsson, and A. Kangas, "System performance evaluation of mobile positioning methods," in *Proc. IEEE Vehicular Technology Conference*, Houston, TX, USA, May 1999.

[7] F. Gustafsson, F. Gunnarsson, N. Bergman, U. Forssell, J. Jansson, R. Karlsson, and P-J. Nordlund, "Particle filters for positioning, navigation and tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, February 2002.

[8] P-J. Nordlund, F. Gunnarsson, and F. Gustafsson, "Particle filters for positioning in wireless networks," in *Invited to EUSIPCO*, Toulouse, France, September 2002.

[9] N.J. Gordon, D.J. Salmond, and A.F.M. Smith, "A novel approach to nonlinear/non-Gaussian Bayesian state estimation," in *IEE Proceedings on Radar and Signal Processing*, 1993, vol. 140, pp. 107–113.

[10] A. Doucet, N. de Freitas, and N. Gordon, Eds., *Sequential Monte Carlo Methods in Practice*, Springer Verlag, 2001.