ANALYSIS OF MISMATCH NOISE IN RANDOMLY INTERLEAVED ADC SYSTEM

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ABSTRACT

Time interleaved A/D converters (ADCs) can be used to increase the sample rate of an ADC system. However, a problem with time interleaved ADCs is that distortion is introduced in the output signal due to various mismatch errors between the ADCs. One way to decrease the impact of the mismatch errors is to introduce additional ADCs in the interleaved structure and randomly select an ADC at each sample instance. The periodicity of the errors is then removed and the spurious distortion is changed to a more noise-like distortion, spread over the whole spectrum. In this paper, a probabilistic model of the randomly interleaved ADC system is presented. The noise spectrum caused by gain errors is also analyzed.

1. INTRODUCTION

The requirements for higher sample rates in A/D converters (ADCs) are ever increasing. To achieve high enough sample rates, a time interleaved ADC system can be used [1, 2], see Figure 1 with $\Delta M = 0$. Here the time interleaved A/D converter works as follows:

- The input signal is connected to all the ADCs.
- Each ADC works with a sampling interval of MT_s, where M is the number of ADCs in the array and T_s is the desired sampling interval.
- The clock signal to the ith ADC is delayed with iT_s. This gives an overall sampling interval of T_s.

The drawback with the interleaved ADC structure is that errors caused by mismatch between the ADCs are introduced:

- Time errors (static jitter)
- Amplitude offset errors
- Gain errors

All these errors distort the sampled signal. For a sinusoidal signal the distortion appears as spurious frequencies in the spectrum. We will only discuss offset and gain errors in this paper.

To avoid the spurious distortion the selection of which ADC that should be used at a certain time instance can be randomized. This means that we pick one ADC at random at each time instance. However, each ADC in the interleaved structure needs M times the desired sampling rate to complete the sampling. Therefore only one ADC is available for selection at each sampling instance. To achieve some randomization one or more extra ADCs must be used [3]. This way there are always at least two ADCs available at each sampling instance, see Figure 1. The randomization spreads the spikes in the spectrum to a more noise-like shape. A probabilistic model of the randomly interleaved ADCs will be presented in Section 3. The spectrum for this kind of ADC system will also be calculated.

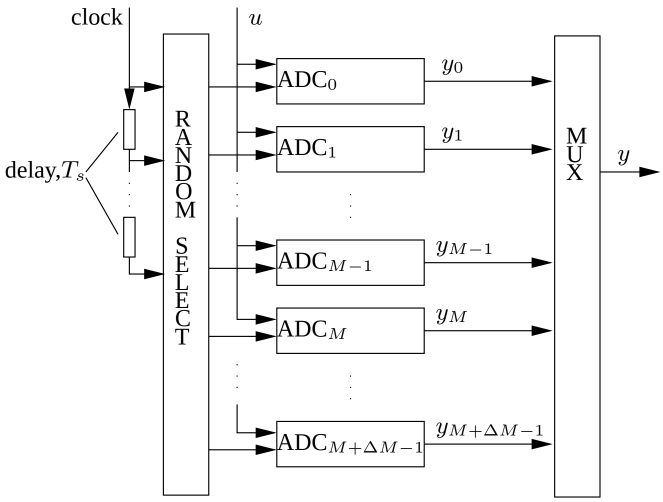

Fig. 1. Random interleaved ADC system with M times higher sampling rate than in each ADC. ΔM additional ADCs are used to achieve some randomization, i.e., $\Delta M + 1$ ADCs are available at each sampling instance.

2. NOTATION AND DEFINITIONS

In this section we introduce the notation used in the rest of the paper. We also define some concepts that are needed in the following. We assume throughout the rest of the paper that the overall sampling interval is one, for the complete ADC system. This assumption is done to simplify notation and is no restriction.

We denote by M the number of ADCs required to achieve the desired sampling rate, where each ADC requires the time MT_s to complete the conversion of one sample. ΔM denotes the number of additional ADCs used to randomize the spectrum. The total number of ADCs in the system are $M + \Delta M$. The gain and amplitude offset errors are denoted $\Delta^0_{g,i}$ and $\Delta^0_{o,i},\ i = 0, \dots, M-1, M, \dots, M-1+\Delta M$ respectively. The sampling instances for each ADC are picked at random and X_k denotes the ADC that is used at time k. The time instances when the ith ADC is used are denoted k_i. We use the following notation for the signals involved:

- $u(t)$ is the analog input signal.
- $u[k]$ is the input signal, sampled without errors.
- $y_i[k_i]$ are the output subsequences from the $M+\Delta M$ ADCs.

$$y_i[k_i] = (1 + \Delta^0_{g,i})u[k_i] + \Delta^0_{o,i} \tag{1}$$

- $y[k]$ is the output signal from the complete ADC system. The subsequences are multiplexed together to form a signal with correct time ordering.

A signal $u[k]$ is quasi-stationary [4] if

$$\bar{E}(u) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E(u[k]) \quad (2)$$

$$R_u[n] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E(u[k+n]u[k]) \quad (3)$$

exist, where the expectation is taken over possible stochastic parts of the signal. A stationary stochastic process is quasi-stationary, with $\bar{E}(u)$ and $R_u[n]$ being the mean value and covariance function respectively.

Assume that $u[k]$ is quasi-stationary. Then the power spectrum of $u[k]$ is defined as [4]:

$$\Phi_u(\omega) = \sum_{n=-\infty}^{\infty} R_u[n]e^{-j\omega n} \quad (4)$$

The next two definitions are used to measure the performance of the ADC system. Assume that the measured signal $y[k]$ consists of a signal part $s[k]$ and a noise and distortion part $e[k]$

$$y[k] = s[k] + e[k] \quad (5)$$

Then the SNDR [5] (Signal to Noise and Distortion Ratio) for $y[k]$ is defined as

$$SNDR = 10\log_{10}\left(\frac{\bar{E}(s^2[k])}{\bar{E}(e^2[k])}\right) \quad (6)$$

Assume that $y[k] = s[k] + e[k]$, $s[k]$ and $e[k]$ are given by

$$s[k] = a\sin(\omega_0 k)$$
$$e[k] = \sum_i b_i \sin(\omega_i k) + \tilde{e}[k] \quad (7)$$

where $\tilde{e}[k]$ is non periodic. Then the SFDR [5] (Spurious Free Dynamic Range) for $y[k]$ is defined as

$$SFDR = 10\log_{10}\left(\frac{a^2}{\max_i b_i^2}\right) \quad (8)$$

3. MISMATCH NOISE

In this section we will calculate the spectrum caused by gain and offset errors in a randomly interleaved ADC. To calculate the spectrum we will also need a probabilistic model of the system, which is also presented in this section.

To calculate the spectrum (4) for $y[k]$, we need the covariance function. Putting (1) into (3) we can calculate the covariance function for $y[k]$, assuming that $\bar{E}(u) = 0$

$$R_y[n] = E\{\Delta^0_{o,X_{k+n}}\Delta^0_{o,X_k}\} + E\{(1+\Delta^0_{g,X_{k+n}})(1+\Delta^0_{g,X_k})\}$$
$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} u(k+n)u(k) = R_{\Delta_o}[n] + R_{\Delta_g}[n]R_u[n] \quad (9)$$

Here $R_{\Delta_o}[n]$ and $R_{\Delta_g}[n]$ depends only on the ADC system, while $R_u[n]$ depends only on the input signal. From (9) we can calculate the spectrum of $y[k]$ as a convolution [6]

$$\Phi_y(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega-\gamma)\Phi_{\Delta_g}(\gamma)d\gamma + \Phi_{\Delta_o}(\omega) \quad (10)$$

To calculate $\Phi_{\Delta_o}(\omega)$ and $\Phi_{\Delta_g}(\omega)$ we need their corresponding covariance functions. The gain error covariance function is

$$R_{\Delta_g}[n] = E\{(1+\Delta^0_{g,X_{k+n}})(1+\Delta^0_{g,X_k})\}$$
$$= P(X_{k+n}=X_k)\frac{1}{M+\Delta M}\sum_{i=0}^{M+\Delta M-1}(1+\Delta^0_{g,i})^2$$
$$+ (1-P(X_{k+n}=X_k))\frac{1}{(M+\Delta M-1)(M+\Delta M)}$$
$$\sum_{i=0}^{M+\Delta M-1}\sum_{j\neq i}(1+\Delta^0_{g,i})(1+\Delta^0_{g,j}) \quad (11)$$

where $P(X_{k+n} = X_k)$ is the probability that $X_{k+n} = X_k$. The calculation of $R_{\Delta_o}[n]$ is identical to (11) with $1+\Delta^0_g$ replaced by Δ^0_o.

To calculate the covariance (11), we need the probability $P(X_{k+n} = X_k)$. To calculate this probability, we first calculate the joint probability over $M-1$ time instances. First, we introduce 2^{M-1} states, represented by binary sequences of length $M-1$, that the ADC system can be in at time $k+n$:

$$00\ldots0 = \{X_{k+n} \neq X_k, \ldots, X_{k+n-(M-2)} \neq X_k\}$$
$$\vdots$$
$$11\ldots1 = \{X_{k+n} = X_k, \ldots, X_{k+n-(M-2)} = X_k\}$$

We assume that $n \geq M-1$. The joint probabilities are denoted

$$P^{(n)}_{a_1a_2\ldots a_{M-1}} = P(a_1a_2\ldots a_{M-1} \text{ at time } k+n),\ a_i \in \{0,1\}$$

This gives a total of 2^{M-1} probabilities. However, since the same ADC cannot be used within a time interval of $M-1$, most of these probabilities are zero

$$P^{(n)}_{a_1a_2\ldots a_{M-1}} = 0, \text{ if } a_i = a_j = 1,\ i\neq j \quad (12)$$

This leaves M probabilities to be calculated

$$P^{(n)} = \begin{bmatrix} P^{(n)}_{10\ldots0} & \cdots & P^{(n)}_{00\ldots1} & P^{(n)}_{00\ldots0}\end{bmatrix}^T \quad (13)$$

These probabilities can be calculated recursively as

$$P^{(n)} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \frac{1}{1+\Delta M} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \frac{\Delta M}{1+\Delta M}\end{bmatrix}}_{A} P^{(n-1)} \quad (14)$$

The assumption at the derivation of these probabilities was that $n \geq M-1$. However, looking into the case $0 < n < M-1$ this can be proved valid for any $n > 0$, and

$$P^{(0)} = \begin{bmatrix} 1 & 0 & \cdots & 0\end{bmatrix}^T \quad (15)$$

The probability $P(X_{k+n} = X_k)$ is equal to $P^{(n)}_{10\ldots0}$. This probability can be calculated from a state space representation, for $n \geq 0$,

$$P^{(n+1)} = AP^{(n)} + B\delta[n+1] \quad (16)$$
$$P(X_{k+n} = X_k) = CP^{(n)}$$

where A is as defined in (14) and

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0\end{bmatrix}^T$$
$$C = \begin{bmatrix} 1 & 0 & \cdots & 0\end{bmatrix}$$

The probability is symmetric in time, so that

$$P(X_{k-n} = X_k) = P(X_{k+n} = X_k) \tag{17}$$

The gain error noise spectrum is calculated from the covariance function, by putting (11) into (4)

$$\Phi_{\Delta_g}(\omega) = \sum_{n=-\infty}^{\infty} R_{\Delta_g}(n)e^{-i\omega n} = \beta_{\Delta_g} \underbrace{\sum_{n=-\infty}^{\infty} e^{-i\omega n}}_{2\pi\delta(\omega)}$$

$$+ \alpha_{\Delta_g} \underbrace{\sum_{n=-\infty}^{\infty} \overbrace{\left(P(X_{k+n} = X_k) - \frac{1}{M+\Delta M}\right)}^{\tilde{R}_\Delta(n)} e^{-i\omega n}}_{\tilde{\Phi}_\Delta(\omega)} \tag{18}$$

Here we have collected the unbounded part in one term, which forms the δ-spike, and the bounded part in one term. The constants α_{Δ_g} and β_{Δ_g} are given by

$$\alpha_{\Delta_g} = \left(\frac{1}{M+\Delta M-1} \sum_{i=0}^{M+\Delta M-1} (1+\Delta_{g,i}^0)^2 \right.$$
$$\left. - \frac{1}{(M+\Delta M-1)(M+\Delta M)} \left(\sum_{i=0}^{M+\Delta M-1} (1+\Delta_{g,i}^0)\right)^2\right)$$
$$\beta_{\Delta_g} = \frac{1}{(M+\Delta M)^2} \left(\sum_{i=0}^{M+\Delta M-1} (1+\Delta_{g,i}^0)\right)^2 \tag{19}$$

The calculations for $\Phi_{\Delta_o}(\omega)$ are similar to (18), with α_{Δ_g} and β_{Δ_g} replaced with α_{Δ_o} and β_{Δ_o}. We get the expressions for α_{Δ_o} and β_{Δ_o} by replacing $1+\Delta_g^0$ with Δ_o^0 in (19).

The convolution with $\delta(\omega)$ gives back the spectrum of the input signal, so the disturbance part comes from $\tilde{\Phi}_\Delta(\omega)$. This part will be evaluated next. By transforming the state space description (16) to a transfer function and eliminating a pole-zero pair in $q = 1$ we get, for $n \geq 0$

$$\tilde{R}_\Delta(n) = P(X_{k+n} = X_k) - \frac{1}{M+\Delta M}$$
$$= \zeta \frac{q^{M-1} + \eta((M-2)q^{M-2} + \cdots + q)}{q^{M-1} + \frac{1}{1+\Delta M}(q^{M-2} + \cdots + 1)} \delta[n] \tag{20}$$

where

$$\zeta = \frac{M+\Delta M-1}{M+\Delta M}, \eta = \frac{1}{M-1+M\Delta M+\Delta M^2}$$

This means that we can calculate the spectrum as

$$\tilde{\Phi}_\Delta(\omega) = \sum_{n=-\infty}^{\infty} \tilde{R}_\Delta(\omega)e^{-i\omega n} \tag{21}$$
$$= -\tilde{R}_\Delta(0) + 2Re\left\{\sum_{n=0}^{\infty} \tilde{R}_\Delta(n)e^{-i\omega n}\right\} =$$
$$\zeta\left(2Re\left\{\frac{e^{i\omega(M-1)} + \eta((M-2)e^{i\omega(M-2)} + \cdots + e^{i\omega})}{e^{i\omega(M-1)} + \frac{1}{1+\Delta M}(e^{i\omega(M-2)} + \cdots + 1)}\right\} - 1\right)$$

3.1. Sinusoidal input

So far we have not assumed anything about the input signal. In this section we evaluate the spectrum with a sinusoidal input. The calculations are done for gain errors, the calculations for amplitude offset errors are similar. We assume here an input signal with spectrum

$$\Phi_u(\omega) = \frac{2\pi A^2}{4}(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)) \tag{22}$$

We then get, from (10), (18) and (22)

$$\Phi_y(\omega) = \frac{A^2\alpha_{\Delta_g}}{4}(\tilde{\Phi}_\Delta(\omega-\omega_0) + \tilde{\Phi}_\Delta(\omega+\omega_0))$$
$$+ \frac{2\pi A^2\beta_{\Delta_g}}{4}(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)) \tag{23}$$

where the second term is the desired signal and the first term is distortion. We will next calculate the SNDR for the output signal. The signal energy is

$$\bar{E}(s^2[k]) = \frac{\pi A^2\beta_{\Delta_g}}{2}\int_{-\pi}^{\pi}(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))d\omega$$
$$= \pi A^2\beta_{\Delta_g} \tag{24}$$

and the distortion energy is

$$\bar{E}(e^2[k]) = \frac{A^2\alpha_{\Delta_g}}{4}\int_{-\pi}^{\pi}(\tilde{\Phi}_\Delta(\omega-\omega_0) + \tilde{\Phi}_\Delta(\omega+\omega_0))d\omega =$$
$$\frac{A^2\alpha_{\Delta_g}}{2}\int_{-\pi}^{\pi}\tilde{\Phi}_\Delta(\omega)d\omega = \pi A^2\alpha_{\Delta_g}\tilde{R}_\Delta[0] = \pi A^2\alpha_{\Delta_g}\zeta \tag{25}$$

Since the gain error spectrum contains no δ-spikes, the SFDR is infinite. We can calculate the SNDR from (6), (24) and (25)

$$SNDR = 10\log_{10}\left(\frac{a}{b-a}\right) \tag{26}$$

where

$$a = \left(\sum_{i=0}^{M+\Delta M-1} (1+\Delta_{g,i}^0)\right)^2$$
$$b = (M+\Delta M)\sum_{i=0}^{M+\Delta M-1} (1+\Delta_{g,i}^0)^2$$

which is the same as for the interleaved ADC without randomization. Assuming that the mean value of the gain errors is zero, and denote the gain error variance

$$\bar{\sigma}_{\Delta_g}^2 = \frac{1}{M+\Delta M}\sum_{i=0}^{M+\Delta M-1} (\Delta_{g,i}^0)^2 \tag{27}$$

we can simplify the SNDR to

$$SNDR = -10\log_{10}\bar{\sigma}_{\Delta_g}^2 \tag{28}$$

3.1.1. Noise comparison

Since we also have quantization noise in the ADC, we do not gain much performance by decreasing the gain error noise below a certain level that depends on the number of bits in the ADC. In this subsection we will compare the noise caused by the gain errors

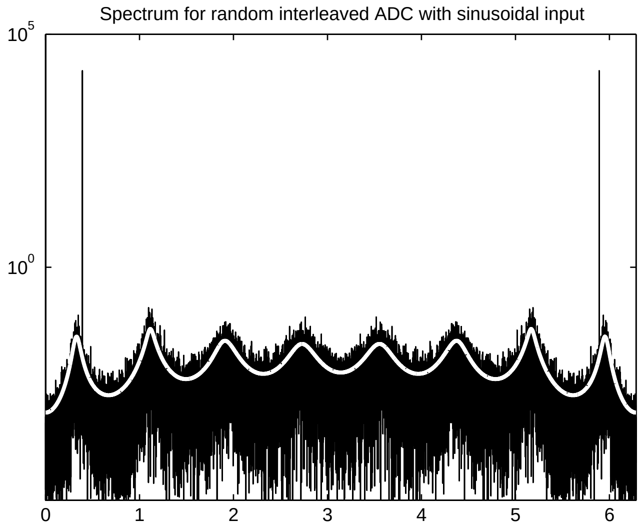

Fig. 2. Output spectrum from random interleaved ADC system with $M = 8$ and $\Delta M = 1$ with sinusoidal input (black curve), together with theoretical gain error noise spectrum for the same input (white curve).

with the quantization noise. The comparison can be done in two ways. The first way is to compare the mean noise, i.e., the SNDR. The other way is to compare the maximum value of the spectra. We will evaluate both cases here. With an N-bit ADC the SNDR (here the SNDR is equal to the SNR) caused by quantization is [5]

$$SNDR_q = 1.76 + 6.02NdB \tag{29}$$

If we compare (28) and (29) and calculate the bound where the gain error noise contributes less than the quantization noise, we get

$$\bar{\sigma}^2_{\Delta_g} < \frac{2}{3}4^{-N} \tag{30}$$

If we instead consider the maximum value of the spectrum, we get

$$\bar{\sigma}^2_{\Delta_g} < \frac{2}{3}4^{-N}\frac{1}{\max_\omega \tilde{\Phi}_{\Delta_g}(\omega)} \tag{31}$$

It is hard to find an analytical solution for the maximum value in general, but for $M = 2$ we have

$$\bar{\sigma}^2_{\Delta_g} < \frac{2}{3}4^{-N}\frac{\Delta M + 1}{\Delta M + 3} \tag{32}$$

4. SIMULATIONS

To evaluate the spectrum calculated in Section 3, we have simulated an A/D converter with $M = 8$ and $\Delta M = 1$ with a sinusoidal input. The spectrum of the output together with the theoretical spectrum is shown in Figure 2. We can see from the figure that the correspondence between simulation and theory is very good. The ideal situation would be that the noise caused by gain errors is white. For white noise the spectrum is constant. To give an idea of how far from white the noise gain error noise is for different values of M and ΔM

$$f(M, \Delta M) = \frac{\max_\omega \Phi_{\Delta_g}(\omega)}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{\Delta_g}(\omega)d\omega} \tag{33}$$

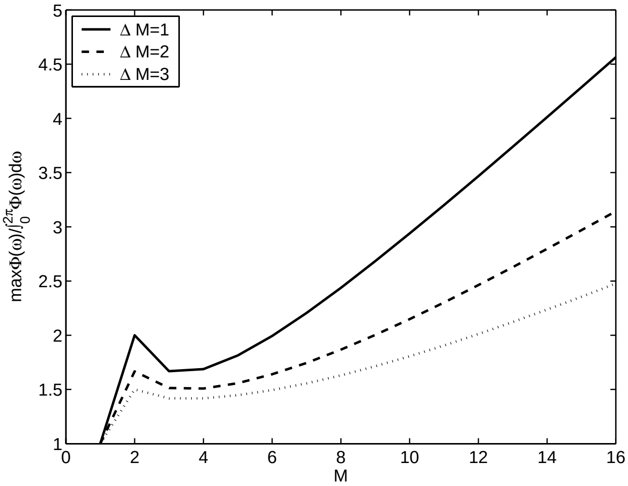

Fig. 3. $f(M, \Delta M) = \frac{\max_\omega \Phi_{\Delta_g}(\omega)}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{\Delta_g}(\omega)d\omega}$ is plotted for M between 1 and 16 and ΔM between 1 and 3, as a measure of how far from white noise the gain error noise is. White noise corresponds to the value 1.

is plotted in Figure 3. For white noise this quotient would be 1. As expected the maximum value of the error spectrum is increased when M is increased, since we have a smaller fraction of ADCs to choose from. However, the peak value for $M = 2$ is a little counter intuitive.

5. CONCLUSION

Various mismatch errors between the ADCs in a time interleaved ADC system, introduce distortion in the output signal. Additional ADCs can be used in the interleaved ADC system to introduce some randomization. By doing this the impact of the mismatch errors is decreased. In this paper we have studied the randomly interleaved ADC system from a probabilistic viewpoint. In Section 3 we have presented a probabilistic model for the ADC system and derived the spectrum caused by gain and offset mismatch. We have also calculated how small the errors must be to give less noise contribution than the quantization for a given number of bits. In Section 4 we have verified the results with simulations of a randomly interleaved ADC system with gain errors.

6. REFERENCES

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