Problem 1.1

Consider the detection problem

$$\begin{aligned} H_0 &: x[0] = w[0] \\ H_1 &: x[0] = 1 + w[0] \end{aligned}$$

where $w[0]$ is zero mean Gaussian random variable with variance σ^2. If the detector decides H_1 if $x[0] > 1/2$, find the probability of making a wrong decision when H_0 is true. To do so, determine the probability of deciding H_1 when H_0 is true or $P_0 = \Pr\{x[0] > 1/2\,;\, H_0\}$. For this to be 10^{-3} what must σ^2 be?

Problem 1.2

Consider the detection problem

$$\begin{aligned} H_0 &: x[0] = w[0] \\ H_1 &: x[0] = 1 + w[0] \end{aligned}$$

where $w[0]$ is uniformly distributed random variable on the interval $[-a,\, a]$ for $a > 1$. Discuss the performance of the detector that decides H_1 if $x[0] > 1/2$ as a increases.

Problem 1.4

In Problem 1.1 now assume that the probability of H_0 being true is 1/2. If we decide H_1 if $x[0] > 1/2$, find the total probability of error or

$$P_e = \Pr\{x[0] > 1/2|H_0\}\,\Pr\{H_0\} + \Pr\{x[0] < 1/2|H_1\}\,\Pr\{H_1\}$$

Plot P_e versus σ^2 and explain your result.

Problem 2.2

Derive (2.4) by nothing that

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}t}\, t\, e^{-\frac{t^2}{2}}\, dt \tag{1}$$

and using integration by parts. Also, explain why the approximation improves as x increases.

Problem 3.1

Determine the NP test for distinguishing between the hypotheses H_0: $\mu = 0$ versus H_1: $\mu = 1$ based on the observed sample $x[0]$ $N(\mu, 1)$. Then, find the Type I error (P_{FA}) and the type II error ($P_M = 1 - P_d$, where P_m is the probability of miss). Finally, plot P_M versus P_{FA}.

Problem 3.8

Find the NP test to distinguish between the hypotheses that a single sample $x[0]$ is observed from the possible PDF's

$$H_0 \quad : p(x[0]) = \frac{1}{2} e^{(-|x[0]|)}$$
$$H_1 \quad : p(x[0]) = \frac{1}{\sqrt{2\pi}} e^{(-x[0]^2/2)}$$

Show the decision regions. Hint: You will need to solve a quadratic inequality

Problem 3.13

Prove that the ROC for the NP detector is a concave function over the interval $[0, 1]$. A concave function is one for which

$$\alpha g(x_1) + (1 - \alpha) g(x_2) < g(\alpha x_1 + (1 - \alpha) x_2)$$

for $0 < \alpha < 1$ and any two points x_1 and x_2. To do so consider two points on the ROC $(p_1, P_D(p_1))$ and $(p_2, P_D(p_2))$ and find P_D for a randomized test. A randomized test first flips a coin with $\Pr\{\text{head}\} = \alpha$. If the outcome is a head, we employ the detector whose performance is $(p_1, P_D(p_1))$. Otherwise, we employ the detector whose performance is $(p_2, P_D(p_2))$. We decide H_1 if the chosen detector decides H_1. Hint: For a given P_{FA} the detection performance of the randomized detector must be less than or equal to that of the NP detector.