Output tracking

2 versions:

1) exact output tracking

2) asymptotic output tracking

Exact output tracking

- also called system inversion
- related (but not identical) to different. flatn.

Assume that instead of $y(t) \equiv 0$ we want the system to follow (exactly) a given output trajectory $y_d(t)$.

-> Must find $x(0)$ and $u(t)$ s.t. the system output coincides with $y_d(t)$.

In the normal form:

$y_d(t) = \xi_1(t) \quad \forall t$ hence for all deriv.

$\xi_2(t) = \dot{y}_d(t)$

$\vdots$

$\xi_r(t) = y_d^{(r-1)}(t)$

$v(t) = y_d^{(r)}(t)$

which in the original basis corresponds to

$$u(t) = \frac{-L_f^r h(\Phi^{-1}(\xi,\eta)) + y_d^{(r)}(t)}{L_g L_f^{r-1} h(\Phi^{-1}(\xi,\eta))}$$

call $\underline{Y}_d(t) \triangleq \begin{bmatrix} y_d(t) \\ \vdots \\ y_d^{(r-1)}(t) \end{bmatrix}$

For the η variables it is then

$$\dot{\eta} = q(\underline{Y}_d(t), \eta) \qquad \eta(0) \text{ still free}$$

$\uparrow$ this is now given, time-dependent profile

Drawback of exact tracking: need to ~~know~~ preset the initial conditions of the system ξ i.e. to fix $\xi(0)$

$\rightarrow$ more realistic to ask to track the output asymptotically

Asymptotic output tracking

Given the system in normal form

$$\dot{\xi} = A\xi + Bv$$
$$\dot{\eta} = q(\xi,\eta)$$
$$y = [1\,0 \cdots 0]\,\xi$$

and a reference output $y_d(t)$

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Consider the output tracking error

$$e(t) = y(t) - y_d(t)$$

exact tracking corresponds to $e(t) \equiv 0 \ \forall t$

Since $\xi_{i+1} = y^{(i)} = L_f^i h(x)$

$$e^{(i)}(t) = y^{(i)} - y_d^{(i)} \overset{!}{=} \overset{= \xi_{i+1} - y_d^{(i)}}{L_f^i h(x) - y_d^{(i)}} \quad i = 0, 1, \ldots, r-1$$

we can replace ξ with the error vector

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_r \end{bmatrix} \triangleq \begin{bmatrix} e \\ \dot{e} \\ \vdots \\ e^{(r-1)} \end{bmatrix} = \begin{bmatrix} \xi_1 - y_d \\ \xi_2 - \dot{y}_d \\ \vdots \\ \xi_r - y_d^{(r-1)} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{\epsilon} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix} \epsilon + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v & \text{new state (error)} \\ \dot{\eta} = q(\xi, \eta) = q(\epsilon + \overline{Y}_d(t), \eta) & \eta \text{ unchanged} \\ e(t) = \epsilon_1(t) & \text{new output (error)} \end{cases}$$

the equiv. of zero dynamics for this system is now when $\epsilon(t) \equiv 0$

i.e. it is $\dot{\eta} = q(\overline{Y}_d(t), \eta)$

this is however time-varying, hence its stability must be in a uniform sense!

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thm Assume $\bar{y}_d(t)$ are defined and bounded $\forall t$ - Let η_d be the solution of

$$\dot{\eta} = q(\bar{y}_d(t), \eta)$$

(i.e. in corresp. of $\epsilon(t) \equiv 0 \rightarrow$ zero dyn.)

Suppose this solution is bounded and uniformly asympt. stable -

Then for $\eta(0)$ near $\eta_d(0)$ and for $\epsilon(0)$ small $\exists$ feedback $v = K\epsilon$ s.t. $A+BK$ Hurwitz is s.t. ~~that~~ $\epsilon(t) \xrightarrow{t \to \infty} 0$ and η bounded -

meaning: bounded tracking is achieved

- $\epsilon(t) \longrightarrow 0$
- η stays bounded

Disturbance decoupling problem

Consider a SISO system with an additive disturbance w

$$\dot{x} = f(x) + g(x)u + p(x)w$$
$$y = h(x)$$

$p(x)$ = vect. field of the disturbance

Problem (disturbance decoupling)
~~Find a fe~~ Under what conditions $\exists$ a feedback law $u = \alpha(x) + \beta(x)v$ s.t. the disturbance w has no effect on the output y ?

thm The disturbance decoupling problem is solvable iff

$$L_p L_f^k h(x) = 0 \quad k = 0, 1, \dots, r-1$$

where r = rel. degree of the system.

Proof revisit the transformation into normal form

$\xi_1 = y = h(x)$ is not affected by w

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$$\dot{y} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u + p(x)w\right)$$

$$= L_f h(x) + \underbrace{L_g h(x) u}_{=0} + \underbrace{L_p h(x) w}_{=0}$$

if $r > 1$

$\Rightarrow$ w does not appear in $\dot{y} = \xi_2$

$\vdots$

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x) u}_{\neq 0} + \underbrace{L_p L_f^{r-1} h(x) w}_{=0}$$

$\Rightarrow$ with the usual choice of input

$$u(x) = \frac{-L_f^r h(x) + v}{L_g L_f^{r-1} h(x)}$$

also $y^{(r)} = \dot{\xi}_r = v$ is not affected by w

$\Rightarrow$ normal form
$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \quad \vdots \\ \dot{\xi}_r = v \\ \dot{\eta} = q(\xi, \eta, w) \\ y = \xi_1 \end{cases}$$

$\Rightarrow$ disturbance is decoupled from output, only zero dynamics is affected (unobservable)

- Meaning: the relative degree $w \to y$ has to be strictly bigger than the rel. degree $u \to y$
- Geometrically:

$$p(x) \perp \text{span}\left\{\frac{\partial h}{\partial x}, \frac{\partial (L_f h)}{\partial x}, \dots, \frac{\partial (L_f^{r-1} h)}{\partial x}\right\}$$

CONTROL LYAPUNOV FUNCTIONS

Consider the system

$$\dot{x} = f(x) + g(x)u \qquad \text{at } x_0 = 0$$

$(\text{s.t. } f(0) = 0)$

Problem: want to find (static) state feedback law $u = \varphi(x)$ s.t. the closed loop
$\dot{x} = f(x) + g(x)\varphi(x)$
is (locally or globally) asymptotically stable at x_0

From a converse Lyapunov thm: if x_0 is (locally or globally) asympt. stable $\Rightarrow$ $\exists$ a Lyapunov function $V > 0$ s.t.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left(f(x) + g(x)\varphi(x)\right) < 0$$

or $\dot{V}(x) < -W(x)$ with $W(x) > 0$ if we want a certain rate of convergence (for instance exponential if $W(x) \sim \|x\|$)

def $V>0$, C^1 is a (local or global) Control Lyapunov Function (CLF) if for some $u \in \mathbb{R}$

$$\frac{\partial V}{\partial x}\left(f(x)+g(x)u\right)<0 \quad \text{for } x \neq 0$$

(locally or globally around $x_0=0$)

Consequences:

1) $\dot{V}<0 \Rightarrow$ when $L_g V(x) = \frac{\partial V}{\partial x} g(x) = 0$
then it must be $L_f V(x) = \frac{\partial V}{\partial x} f(x) < 0$

meaning: drift must be asymptotically stable when the control part cannot compensate for it

2) If we want $u=\psi(x)$ continuous (so that the closed-loop sys is continuous) $\psi(0)=0$, then V has the small control prop.

def V has the small control property if $\forall \epsilon>0\ \exists \delta>0$ s.t. $\forall x \neq 0$ and $\|x\|<\delta$ then $\exists u$ with $\|u\|<\epsilon$ s.t. $L_f V(x) + L_g V(x) u < 0$

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meaning $\mathring{V} < 0$ cannot be guaranteed by blowing the control amplitude when $\|x\| \to 0$ (If so you get discontinuous behavior at $x=0$).

- How to choose $u = \psi(x)$?
  For instance try "canceling the dynamics"

$$u = \psi(x) = \begin{cases} \dfrac{-L_f V(x) + \gamma(x)}{L_g V(x)} & \text{when } L_g V(x) \neq 0 \\ \text{anything} & \text{when } L_g V(x) = 0 \end{cases}$$

$$\Rightarrow \mathring{V} = \cancel{L_f V(x)} + \cancel{L_g V(x)} \left(\frac{-\cancel{L_f V(x)} + \gamma(x)}{\cancel{L_g V(x)}} \right)$$

$$= \gamma(x)$$

for $L_g V(x) \neq 0$

Difficulty: choose $\gamma(x)$ so that $u = \psi(x)$ is continuous and $\mathring{V}(x) < 0$ $\forall x$
($\gamma(x)$ must be < 0 for $L_g V(x) \neq 0$)

when the small control property holds then there is a constructive formula, called <u>universal Sontag formula</u>

<u>thm</u> Let V be a (local or global) CLF and assume the small control property holds. Then the origin is (locally or globally) stabilizable by the feedback $u = \varphi(x)$ where

$$\varphi(x) = \begin{cases} -\dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^4}}{L_g V(x)} & \text{if } L_g V(x) \neq 0 \\ 0 & \text{if } L_g V(x) = 0 \end{cases}$$

<u>Proof</u>

- If $L_g V(x) = 0 \Rightarrow \dot{V} = L_f V(x) < 0$ for $x \neq 0$ by def. of CLF
- If $L_g V(x) \neq 0$

$$\dot{V} = \cancel{L_f V(x)} - \cancel{L_g V(x)} \frac{\left(\cancel{L_f V(x)} + \sqrt{(L_f V)^2 + (L_g V)^4}\right)}{\cancel{L_g V(x)}}$$

$$= -\sqrt{(L_f V)^2 + (L_g V(x))^4} < 0$$

- If $L_g V(x) = 0$ then $\dot{V} = L_f V < 0$ by def of CLF n213

- consequence: $\exists$ of a CLF is necessary and sufficient for (local or global) stabilizab.

- Difficulty: a CLF must be given!
- How to find a CLF? If you have any method that stabilizes the system then you ~~hav~~ also get a CLF (but it could not be easy to find V for a as. stable closed loop system ...)

example:
$$\dot{x} = ax - bx^3 + u \qquad a, b > 0$$
from feedback linearization:
$$u = -ax + bx^3 - kx \qquad k > 0$$
leads to the closed loop system
$$\dot{x} = -kx$$
which is globally as. (exp.) stable and has $V(x) = \frac{1}{2}x^2$ as Lyapunov funct. (radially unbounded)

let us compute ~~Sontag~~ Sontag formula:

$$L_g V(x) = \frac{\partial V(x)}{\partial x} g = x$$

$$L_f V(x) = \frac{\partial V}{\partial x} f = x(ax - bx^3)$$

in x s.t. $L_g V \neq 0$ (i.e. for $x \neq 0$)

$$\Rightarrow u = \varphi(x) = -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V}$$

$$= -\frac{\cancel{x}(ax - bx^3) + \sqrt{x^2(ax-bx^3)^2 + x^4}}{\cancel{x}}$$

$$= -ax + bx^3 - x\sqrt{(a-bx^2)^2 + 1}$$

compare this feedback with the feedback linearizing u

$$u = -ax + bx^3 - kx$$

//

- Sontag formula has a robustness w.r.t. the input amplitude: replacing $u = \varphi(x)$ with $u = k\varphi(x)$ $k > 1/2$ stabilization property is preserved.
- meaning of the formula: it guarantees sufficient smoothness to $\varphi(x)$