Special case: when relative degree is $r = n$

$\Rightarrow y^{(n)} = v$ or the T.F. $Y(s) = \frac{1}{s^n} V(s)$

$\Rightarrow$ chain of n integrators

$\Rightarrow$ entire state space become

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = L_f^n h(x) + L_g L_f^{n-1} h(x) u = v \quad \text{if we choose } u = \dfrac{-L_f^n h(x) + v}{L_g L_f^{n-1} h(x)} \\ y = \xi_1 \end{cases}$$

$\Rightarrow$ entire state space is linearized

$\Rightarrow$ feedback linearization!

In this way the change of state and change of input are known explicitly

$$\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \qquad u = \frac{-L_f^n h(x) + v}{L_g L_f^{n-1} h(x)}$$

$\downarrow$

to show that this is a good diffeom., use that $\frac{\partial h(x)}{\partial x}, \frac{\partial}{\partial x} L_f h(x), \ldots, \frac{\partial}{\partial x} L_f^{n-1} h(x)$ are all lin. indep.

$\Rightarrow \frac{\partial \Phi}{\partial x}$ is full rank.

In this way it is possible to obtain simpler conditions for feedback linearizability:

<u>thm</u> The SISO system $\dot{x} = f(x) + g(x)u$ with $f(x_0) = 0$ is feedback linearizable at x_0 iff

1) $\dim(\mathrm{span}\{ad_f^k g,\ k = 0, 1, \dots, n-1\}) = n \quad \forall x \in B(x_0)$

2) the distrib $D = \mathrm{span}\{ad_f^k g \quad k = 0, 1, \dots, n-2\}$ is involutive $\forall x \in B(x_0)$

to prove this need the following

-> this requires $[g, ad_f g]$, etc to be lin dep => not the same as cond 1)

<u>Prop</u> $L_g L_f^k h(x) = 0 \quad \forall k = 0, 1, \dots r$

$\iff$

$L_{ad_f^k g} h(x) = 0 \quad \forall k = 0, 1, \dots, r$

(valid also if $r < n$)

first <u>consequence</u> (another <u>prop</u>): $\frac{\partial}{\partial x} h(x), \frac{\partial}{\partial x} L_f h, \dots, \frac{\partial}{\partial x} L_f^{n-1} h$ lin indep $\iff$ $g, ad_f g, \dots, ad_f^{n-1} g$ are lin indep $\Rightarrow$ 1) of the thm follows meaning of the proposition: $L_g L_f^k h(x)$ is a k-th order PDE in $h(x)$

ex $L_g L_f h(x) = \frac{\partial (L_f h(x))}{\partial x} g = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) g$ 2nd ord. PDE

while $L_{ad_f^k} h(x)$ is a 1st order PDE in $h(x)$

ex: $L_{ad_f g} h(x) = \frac{\partial h(x)}{\partial x} \underbrace{[f, g]}_{\text{this is a v.f.}}$ 1st order PDE in $h(x)$

Now $L_{ad_f^k g} h(x) = 0 \quad \forall k = 0, 1, \dots, n-2$

means $\frac{\partial h}{\partial x} ad_f^k g = 0 \quad \forall k = 0, 1, \dots, n-2$

i.e. $\frac{\partial h}{\partial x} \perp \text{span}\{g, ad_f g, \dots, ad_f^{n-2} g\}$

which is the involutivity cond 2) of the thm.

Since $\frac{\partial h}{\partial x}$ is the differential ("gradient") of the function h (Frobenius thm) //

Notice that the two cond. of the thm do not depend on h.

If they are satisfied then it is possible to find the output $h(x)$ which leads to relative degree n (maybe a "dummy output") by solving the 1st order PDEs in $h(x)$ (unknown)

$$L_g h(x) = 0 \qquad\qquad \frac{\partial h}{\partial x} g = 0$$

i.e.

$$L_{ad_f g} h(x) = 0 \qquad\qquad \frac{\partial h}{\partial x} [f, g] = 0$$

$$\vdots \qquad\qquad \vdots$$

$$L_{ad_f^{n-1} g} h(x) = 0 \qquad\qquad \frac{\partial h}{\partial x} ad_f^{n-1} g = 0$$

<u>example</u> (of last time)

$$\begin{cases} \dot{x}_1 = \gamma \sin x_2 \\ \dot{x}_2 = -x_1^2 + u \end{cases}$$

we know this is feedback linearizable by change of ~~feedback~~ input $u = x_1^2 + \dfrac{v}{\gamma \cos x_2}$

and change of state $\Phi(x) = \begin{bmatrix} x_1 \\ \gamma \sin x_2 \end{bmatrix}$

But we also know that if the output is $y = x_2$, then the relative degree is $r = 1$

$$\dot y = \frac{\partial h}{\partial x}\dot x = \frac{\partial h}{\partial x}\left(f(x)+g(x)u\right) = L_f h(x) + L_g h(x) u$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} \gamma \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\right)$$

$\Rightarrow L_g h(x) \neq 0 \quad \Rightarrow r = 1$

Let us check the new conditions for feedb. lin.

$$\text{span}\{g, ad_f g\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \gamma \cos x_2 \\ 0 \end{bmatrix}\right\}$$

has dim 2 near $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ cond 1) is ok

$D = \text{span}\{ad_f^k g \quad k = 0, 1, \ldots, n-2\} = \text{span}\{g\}$, since $n-2=0$

$\Rightarrow$ involutive by construction $\Rightarrow$ cond 2) ok

$\Rightarrow$ indeed feedback lineariz.

To find $h(x)$ that leads to $r = 2$

$$\frac{\partial h(x)}{\partial x} g = 0 \quad \text{i.e.} \quad \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

i.e. $\frac{\partial h(x)}{\partial x_2} = 0 \quad \Rightarrow \quad h(x) = h(x_1)$ any such funct. is ok

for instance

$$h(x) = x_1$$

$\Rightarrow L_f h(x) = \frac{\partial h}{\partial x} f = [1 \; 0] \begin{bmatrix} \gamma \sin x_2 \\ -x_1^2 \end{bmatrix} = \gamma \sin x_2$

$\Rightarrow \Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ \gamma \sin x_2 \end{bmatrix}$ same Φ we used before! but now $y = h(x)$ is different.

$\Rightarrow \begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = v \\ y = \xi_1 \end{cases}$ double integrator

$\Rightarrow$ changing output also the relative degree ~~can~~ change!

Zero dynamics

Let us return to the normal form for a SISO system of rel. degree $r \leq n$

$$\begin{cases} \dot{\xi} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & \cdots & & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \\ \dot{\eta} = q(\xi, \eta) \\ y = [1\,0 \cdots 0]\,\xi \end{cases}$$

We want to study the problem of zeroing the output (and maintaining it to 0)

$$y(t) \equiv 0 \quad \forall t \quad \Rightarrow \quad y^{(k)}(t) \equiv 0 \quad \forall t$$

since $y^{(k)} = \xi_{k+1}$ $\Rightarrow$ $\begin{cases} \xi_k(t) \equiv 0 & k = 1, \dots, r \\ v(t) \equiv 0 \end{cases}$

while for the η subsystem we have

$$\dot{\eta} = q(0, \eta)$$

for any initial cond. $\eta(0)$.

def The subsystem $\dot{\eta} = q(0, \eta)$ is called the <u>zero dynamics</u> of the system the zero dynamics represents the "internal dynamics" of the system, compatible with a zero output $y(t) \equiv 0$

Origin of the name: from "transmission zeros" of a linear system (i.e. $n-r$ roots of the numerator of a transf. funct. of rel. degree r and denomin. of order n)

If $\dot{\eta} = q(0, \eta)$ linear then its eigenvalues are exactly the zeros of the T.F.

In the original basis x, the output is zeroed on the manifold

$$Z^* = \{ x \text{ s.t. } h(x) = 0, L_f h(x) = 0, \dots, L_f^{r-1} h(x) = 0 \}$$

called <u>output-zeroing manifold</u>

When $v = 0$ i.e. $u = -\dfrac{L_f^r h(x)}{L_g L_f^{r-1} h(x)}$ then

Z^* is an invariant manifold for the dynamics (like those we saw in center manif. th.)

i.e. $x(0) \in Z^* \Rightarrow x(t) \in Z^* \quad \forall t$

The dynamics on this manifold is the zero dynamics - (zero dynamics is difficult to express in the original basis).

With this choice of u, the zero dynamics is isolated from both input and output clearly in order to have a reasonable behavior we need that this zero dynamics does not blow (i.e. that it is stable).

def the system $\dot{x} = f(x) + g(x)u$ is locally asymptotically (resp. exponentially) <u>minimum phase</u> at x_0 (s.t. $f(x_0)=0$) if the equil point $\eta = 0$ of $\dot{\eta} = q(0, \eta)$ is locally asymptotically (resp. expon.) stable.

Terminology "minimum phase" is adopted from the minimum phase of a linear system (i.e. zeros of the T.F. have $Re[\lambda] < 0$)

<u>example</u> (earlier)

$$\begin{cases} \dot{x} = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} u \qquad x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ y = x_3 \end{cases}$$

we computed rel. degr. $r = 2$

normal form:

$$\Phi(x) = \begin{bmatrix} x_3 \\ x_2 \\ 1 + x_1 - e^{x_2} \end{bmatrix}$$

i.e. $\Phi(0) = 0$

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = v$$
$$\dot{\eta} = (1 - \eta - e^{\xi_2})(1 + \xi_2 e^{\xi_2})$$

on Z^*: $\begin{cases} \xi_1(0) = \xi_2(0) = 0 \\ v = 0 \end{cases}$

$\Rightarrow \dot{\eta} = (1 - \eta - 1)(1 - 0) = -\eta$

$\Rightarrow$ zero dynamics is (globally) asympt. stable at $\eta_0 = 0$ (even exponent.)

$\Rightarrow$ system is minimum phase (exponentially so) n200

Local asymptotic stabilization

When the zero dynamics is stable then the entire system can be rendered stable by feedback

Consider the system in normal form:

$$\begin{cases} \dot{\xi} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & & \cdots & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v = A\xi + Bv \\ \dot{\eta} = q(\xi, \eta) \\ y = [1\ 0 \cdots 0]\,\xi = C\xi \end{cases}$$

Assume $\eta_0 = 0$ is an equil point of the zero dynamics

__Thm__ If the zero dynamics is locally asympt. stable, then ~~the~~ the feedback $v = K\xi$ s.t.

$$(A+BK) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ k_1 & \cdots & & k_r \end{bmatrix} \text{ is Hurwitz}$$

locally asympt. stabilizes the equil $\begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of the entire system.

n201

Proof (idea) Use a converse Lyapunov thm to say that asympt. stabil. of $\eta \Rightarrow \exists$ Lyapun. function $V_1(\eta)$ for η s.t.

$$\frac{\partial V_1}{\partial \eta} q(0,\eta) \leq -\alpha(\|\eta\|) \quad \alpha \in K \text{ (class K)}$$

Since $A+BK$ Hurwitz $\Rightarrow \exists\ P=P^T>0$ which solves the Lyapunov eq with $Q=+I$

$$(A+BK)^T P + P(A+BK) = -I < 0$$

Use the candidate Lyapunov function

$$V(\xi,\eta) = V_1(\eta) + k\sqrt{\xi^T P \xi}$$

(square root: I want it first order)

$$\Rightarrow \dot V = \frac{\partial V_1}{\partial \eta} \underbrace{q(\xi,\eta)}_{\substack{\text{add and subtract} \\ q(0,\eta)}} + \frac{k}{\sqrt{\xi^T P \xi}} \xi^T \underbrace{\left((A+BK)^T P + P(A+BK)\right)}_{=-I} \xi$$

$$= \frac{\partial V_1}{\partial \eta} q(0,\eta) + \underbrace{\frac{\partial V_1}{\partial \eta}\left(q(\xi,\eta) - q(0,\eta)\right)}_{\substack{\text{term which can be} \\ \text{positive} \Rightarrow \text{must be} \\ \text{bounded above} \\ \leq \gamma_1 \|\xi\|}} - \underbrace{\frac{k\,\xi^T \xi}{\sqrt{\xi^T P \xi}}}_{\leq k\gamma_2 \|\xi\|}$$

near the origin:

h202

$\Rightarrow \dot{V} \leq -\alpha(\|\alpha\|) + \gamma_1 \|\xi\| - \gamma_2 k \|\xi\|$

$\Rightarrow$ for $k > \frac{\gamma_1}{\gamma_2}$ $\quad \gamma_1 - \gamma_2 k < 0$

$\Rightarrow \dot{V} < 0 \Rightarrow$ local asympt. stab. //

If $\dot{\eta} = q(0,\eta)$ is (locally) exp. stable then also lineariz. can be used in the proof.

Overall feedback in the original basis is

$$u = \frac{1}{L_g L_f^{r-1} h(x)}\left(-L_f^r h(x) + [k_1 \dots k_r]\begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x)\end{bmatrix}\right)$$

<u>example</u> (last example) $\begin{cases} \dot{x} = \begin{bmatrix} -x_1 \\ 2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} u \\ y = x_3 \end{cases}$ $\quad r = 2$

we saw $\dot{\eta} = -\eta \Rightarrow$ syst. is globally exponentially minimum phase

Any $k_1 < 0$, $k_2 < 0$ leads to $A + BK = \begin{bmatrix} 0 & 1 \\ +k_1 & k_2 \end{bmatrix}$ Hurwitz

$\Rightarrow$ also ξ subsyst is globally asympt. st.

$$\Rightarrow u = \frac{1}{L_g L_f h(x)}\left(-L_f^2 h(x) + [k_1 \ k_2]\begin{bmatrix} h(x) \\ L_f h(x)\end{bmatrix}\right) = -x_1 x_2 + [k_1 k_2]\begin{bmatrix} x_3 \\ x_2 \end{bmatrix}$$

in the original basis

$\Rightarrow$ global asympt. stabilization in the original basis