<u>example</u> linear system $\dot{x} = Ax + Bu$

where $B = [b_1 \ldots b_m]$

controllability => stabilizability

=> realization theory

=> LQ optimal control.

controllab <=> Kalman condition

$$\text{rank}[B\ AB \ldots A^{n-1}B] = n$$

enough to stop at n-1 by Cayley-Hamilton thm

meaning:

$\text{Im}[B\ AB \ldots A^{n-1}B] = \mathcal{R}$ = reachable space

which is a vector (sub) space

If Kalman condit. is satisfied $\mathcal{R} = \mathbb{R}^n$

i.e. each point in the state space can be reached

For a nonlinear system: controllability by linearization is the simplest approach

at x_0 s.t. $f(x_0) = 0$ (equil for the unforced system, $u_i = 0$ ~~(also initial cond~~)

compute Jacobian lineariz.

$$\dot{x} = \frac{\partial f(x_0)}{\partial x} x + \sum \underbrace{g_i(x_0)}_{} u_i = A_0 x + B_0 u$$

these are now constants, not v.f.

$$A_0 = \frac{\partial f(x_0)}{\partial x} \qquad B_0 = \left[g_1(x_0) \dots g_m(x_0) \right]$$

If rank $[B_0 \; A_0 B_0 \dots A_0^{n-1} B_0] = n \Rightarrow$ original nonlinear system is controllable locally around x_0 i.e. $\exists$ open neighborhood $\Omega \subset M$ of x_0 ~~s.t.~~ which can be reached in time T by choosing some $u \in \mathcal{U}$

- controllability by linearization is often unsatisfactory

<u>example</u> rolling wheel

$$\dot{x} = g_1(x)\,v + g_2(x)\,\omega = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

driftless nonlinear underactuated system

$m = 2$, $n = 3$

- equil $\bar{x} = 0$
- Jacobian lineariz : $A_0 = 0$ (no drift)

$$B_0 = [g_1(0), g_2(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad u = \begin{bmatrix} v \\ w \end{bmatrix}$

$\Rightarrow$ rank $[B_0, A_0 B, \dots A_0^{n-1} B_0] =$ rank $[B_0] = 2$

$\Rightarrow$ Kalman condition is not satisfied

$\Rightarrow$ lineariz is not controllable

however experience tells us that the rolling wheel <u>is</u> controllable

$\Rightarrow$ nonlinear controllability

Reachable sets

Consider again $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (*)$

Given $x_0 \in M$

def $\mathcal{R}(x_0, T)$ = reachable set at time $T \geq 0$

$= \{ x(T) \in M$ s.t. $\exists$ admissible input $u : [0,T] \to U$, $u \in \mathcal{U}$ s.t. it is $x(0) = x_0$

$x(T) = \Phi_t^{f + \sum g_i u_i}(x_0, u) \}$ i.e. the solution of $(*)$ with input u is $x(T)$ at time T

def $\mathcal{R}(x_0, \leq T)$ = reachable set in time not greater than T

$= \bigcup_{t \leq T} \mathcal{R}(x_0, t)$

Small-time local controllability (STLC)

def $(*)$ is STLC if $\mathcal{R}(x_0 \leq T)$ contains open neighborhoods of x_0 $\forall T > 0$

i.e. if
- $\mathcal{R}(x_0, \leq T)$ contains open neigh. of M
- $x_0 \in \operatorname{int}(\mathcal{R}(x_0, \leq T))$ $\forall T$

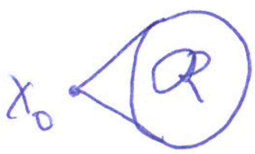
meaning of STLC: the system is STLC at x_0 if we can reach nearby points in arbitrary small times while staying near x_0 for all time.

Accessibility

def $(*)$ is *locally accessible* from x_0 if $\mathcal{R}(x_0, \leq T)$ contains nonempty open sets of M $\forall T > 0$

difference between Accessibility and STLC:

accessib: $\begin{cases} \mathcal{R}(x_0, \leq T) \text{ is an open set in } M \\ \not\Rightarrow x_0 \in \operatorname{int}(\mathcal{R}(x_0, \leq T)) \end{cases}$



STLC: open sets must "grow" around x_0

Accessibility is a less useful property but it is easier to verify, as it admits an <u>infinitesimal characterization</u> in terms of the rank of a Lie algebra of v.f. at the point x_0

<u>def</u> the <u>accessibility algebra</u> is the smallest Lie algebra containing $f, g_1, \dots, g_m$

$$C = \mathrm{Lie}\{f, g_1, \dots g_m\} \subseteq \mathcal{V}(M)$$

<u>thm</u> (<u>Lie Algebraic Rank Condition - LARC</u>)

If $\dim C(x_0) = n \Rightarrow \mathcal{R}(x_0, \leq T)$ contains a nonempty open set of M (i.e. accessibility property holds)

<u>proof</u> (idea)

If $\dim C(x_0) = n \Rightarrow$ by continuity $\dim C(x) = n$ $\forall x \in B(x_0, \epsilon) \Rightarrow \exists\, n$ linearly indep. v.f. $l_1, \dots l_n \in \mathcal{V}(M)$ s.t. their flow compos.

$$\Phi^{l_n}_{t_n} \circ \Phi^{l_{n-1}}_{t_{n-1}} \circ \dots \circ \Phi^{l_1}_{t_1}(x_0)$$ has full rank for small t_i

i.e. it can cover open sets in M //

LARC $\Rightarrow$ local accessibility

LARC is an infinitesimal condition, on the tang. sp. $T_{x_0} M$ (which is a <u>linear</u> v. sp.) $\Rightarrow$ condition is easy to verify!

$\dim C(x_0) = n \Leftrightarrow C(x_0) = T_{x_0} M$ since they are v. spaces.

Notice that flows used in the proof are not "quadrilateral" (as in the Lie bracket flow composition) $\Rightarrow$ we are maybe flowing only with "positive times" (typically $e_1 = f = \text{drift}$ which cannot be reversed)

$\Rightarrow$ Accessibility $\neq$ controllability

<u>example</u> $\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases}$ $f(x) = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}$ $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dot{x} = f(x) + g(x) u$

linearization: at $\begin{cases} x_2 = 0 \\ x_1 \text{ any} \end{cases}$ $A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ inconclusive

Compute Lie brackets

$$[f,g] = \cancelto{0}{\frac{\partial g}{\partial x}} f - \frac{\partial f}{\partial x} g = 0 - \begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ 0 \end{bmatrix}$$

$$[[f,g],g] = \cancelto{0}{\frac{\partial g}{\partial x}} [f,g] - \frac{\partial [f,g]}{\partial x} f = 0 - \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\Delta = \text{span}\{f,g\}$ has dim 2 for $x_2 \neq 0$

$\text{span}\{f, g, [f,g]\}$ " " " " $x_2 \neq 0$

$\text{span}\{f, g, [f,g], [[f,g],g]\}$ has dim 2 everywhere

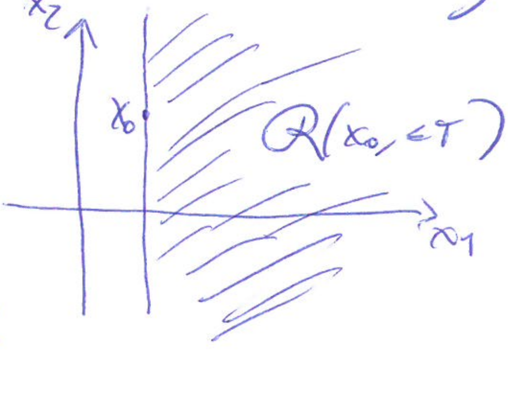
$\Rightarrow C = \text{Lie}\{f,g\} = \text{span}\{f, g, [f,g], [[f,g],g]\}$
= accessibility algebra

$2 = \dim T_x M \quad \forall x \Rightarrow$ syst has LARC and hence it has accessibility everywhere.

However, system is not controllable, since
$\dot{x}_1 = x_2^2 \geq 0 \Rightarrow x_1$ is always non-decreasing

taking any x_0

$x_0 \notin \text{int}(R(x_0, \leq T))$



for any $T > 0$

LARC $\not\Rightarrow$ STLC

<u>example</u> linear system: apply nonlin controllab/access. notions

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i = Ax + Bu \qquad B = [b_1 \dots b_m]$$

v.f.: $f(x) = Ax$ linear

$g_i(x) = b_i$ const.

Assume Kalman cond. holds (i.e. we know it is controllable)

$\mathrm{Im}[B\ AB \dots A^{n-1}B] = \mathbb{R}^n$

$$\Delta(x) = \mathrm{span}\{Ax, b_1 \dots, b_m\}$$

compute accessibility algebra

$$[Ax, b_i] = -Ab_i \qquad [b_i\ b_j] = 0$$

$$[Ax, [Ax, b_i]] = A^2 b_i \quad \text{and so on}$$

$$[Ax, [Ax, [Ax, b_i]]] = -A^3 b_i \quad !$$

accessib. Lie algebra at x_0:

$$C(x_0) = \mathrm{span}\{Ax_0, b_i, Ab_i, A^2 b_i, \dots A^{n-1} b_i \quad i = 1, \dots m\}$$

$$\overset{!}{=} \mathrm{span}\{Ax_0\} + \mathrm{Im}[\underbrace{B\ AB \dots A^{n-1}B}_{\text{Kalman matrix}}]$$

At $x_0 = 0$ LARC $\equiv$ Kalman condition

We know that LARC in general guarantees only accessibility, not controllability (linear syst. are exceptional in this respect)

at $x_0 = 0$ $\mathrm{rank}[B\ AB \dots A^{n-1}B] = n \Rightarrow \dim C(0) = n$

at any x_0 : even when controllab is missing but $\text{rank}[B\ AB \ldots A^{n-1}B] = n-1$

the linear system can still be accessible thanks to $\text{span}\{Ax\}$

$$C(x) = \text{span}\{Ax\} + \text{Im}\{B\ AB \ldots A^{n-1}B\}$$

means it can still be $\dim C(x) = n$ even in this case of $\text{rank}[B\ AB \ldots A^{n-1}B] = n-1$

ex Jacobian lineariz.

$$\dot{x} = f(x) + \sum g_i(x) u_i \qquad \cancel{x_0}\ \text{s.t. } f(x_0) = 0$$

compute $A_0 = \dfrac{\partial f(x_0)}{\partial x} \qquad B_0 = [g_1(x_0) \ldots g_n(x_0)]$

$$\dot{x} = A_0 x + B_0 u$$

condition we just obtained

$$C(x_0) \supseteq \text{Im}[B_0\ A_0B_0 \ldots A_0^{n-1}B_0]$$

~~must~~ must hold also locally for the Jacobian linearization (and implies that controllab of the linearz. ⇒ STLC of the nonlinear system)

This result should hold also for the original nonlinear system if we use the same type of Lie brackets used for the linearization these brackets are

$$[A_0x, B_0],\ [A_0x, [A_0x, B_0]],\ [A_0x, [A_0x, [A_0x, B_0]]] \ldots$$

for the nonlin system these are

$$[f, g],\ [f, [f, g]],\ [f, [f, [f, g]]], \ldots$$

i.e. "adjoint"-type of Lie brackets

$$ad_f^k g$$

<u>thm</u>

scalar nonlinear syst: $\dot{x} = f(x) + g(x)u$

x_0 s.t. $f(x_0) = 0$

Assume LARC is achieved only through "ad-commutators" i.e.

$$\dim(\text{span}\{f, g, ad_f^k g, k = 1, 2, \ldots\}) = n$$

Then at x_0, $R(x_0, T)$ contains a full neighborhood of x_0 (i.e. STLC holds)

→ special case in which LARC ⇒ STLC

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Suff. conditions for STLC

scalar system $\dot{x} = f(x) + g(x)u$

denote $\mathcal{S}^1 = \text{span}\{f, g, [f,g]\}$

$\mathcal{S}^r = \text{span}\{f, g, [f,g], \underbrace{[f,[g,[f,[g \dots]]]]}\}$

~~at most~~ at most ~~exactly~~ r-times g and arbitrarily many f

<u>thm</u> Consider x_0 s.t. $f(x_0) = 0$

If $\mathcal{S}^r(x_0) = T_{x_0}M$ for some r

and $\mathcal{S}^{2q}(x_0) = \mathcal{S}^{2q+1}(x_0)$ $\forall q$

then the system is STLC at x_0

<u>meaning</u>: obstructions to STLC come from $\mathcal{S}^{2q}$ with $2q$ = even (i.e. with an even number of v.f. g appearing $\to$ "bad" brackets)

$\Rightarrow$ 1) need to fill up all directions of $T_{x_0}M$ with only "good" brackets (those containing an odd n° of g)

2) need to "neutralize" the effect of the bad brackets (i.e. write them as linear combinations of brackets with a lower number of g)

STLC: you can cover a neighb. of x_0 while staying near x_0 for all times

In some cases you can have controllability (for times long enough) but not STLC

<u>example</u> bilinear system on S^2

$$\dot{x} \overset{!}{=} A_3 x + A_2 x u \qquad \left(= f(x) + g(x) u\right)$$

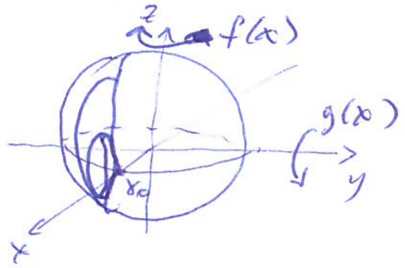
$A_2, A_3 \in so(3)$

$\Rightarrow x \in S^2$ (we have already seen this)

From previous lecture:

$\Delta = span\{f, g\} = \bar{\Delta} = C$ accessib. subalg.

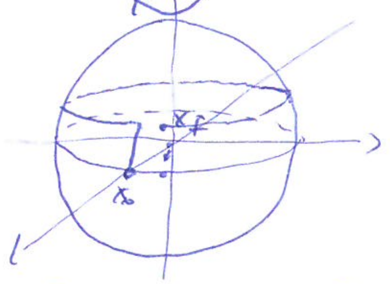
Is the system STLC? no!



$R(x_0, \leq T)$ are open sets in S^2 but do not contain x_0

$x_0 \notin int(R(x_0, \leq T))$ $\forall T \geq 0$
T suff. short!

However, if T is sufficiently long, the system becomes controllable thanks to the drift $f(x) = A_3 x$



use $g(x)$ to go the correct latitude (i.e. parallel) and then just let the drift move x to the correct longitude (i.e. meridian)

for <u>compact</u> manifolds: controllability (for times long enough) is easier than non-compact manif

Controllability of driftless systems

(*) $\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$

driftless: when $u_i = 0$ $\dot{x} = 0$ i.e. drift-free

⇒ simpler because we can "reverse" any direction of motion

⇒ intuitively clear that accessibility collapses into controllability

Take $\Delta = \text{span}\{g_1, \dots, g_m\}$
compute involutive closure
$$\bar{\Delta} = \text{Lie}\{g_1, \dots g_m\} = \mathcal{C} = \text{accessibility Lie alg.}$$
(= control Lie alg.)

<u>thm</u> (<u>Chow</u>)

the system (*) is STLC at $x_0 \in M$ if
$\bar{\Delta}(x_0) = T_{x_0} M$

meaning: if $\bar{\Delta}(x_0) = T_{x_0} M$ ⇒ $\dim(\bar{\Delta}(x_0)) = \dim(T_{x_0} M) = \dim(M)$

⇒ LARC holds ⇒ $\mathcal{R}(x_0, \leq T)$ contains open neigh. of x_0, since all v.f. of $\bar{\Delta}$ can be reversed.

example rolling wheel

$$\dot{z} = g_1(z)v + g_2(z)\omega = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega \quad \text{driftless}$$

$$[g_1\ g_2] = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = g_3(z)$$

$\bar{\Delta} = \operatorname{span}\{g_1, g_2, g_3\}$ has $\dim 3 = \dim M$

$\Rightarrow$ Chow theorem holds

$\Rightarrow$ LARC holds

$\Rightarrow$ wheel is STLC

//

NONHOLONOMIC SYSTEMS

What is it that makes Δ non-integrable?
It is the non-commutativity of the v.f.
(i.e. the Lie-bracket that generate new directions)

Just like there are vectors and their duals that are co-vectors i.e. objects that map vectors into real numbers, and that are in the orthogonal complement of a vector subspace (they annihilate all vectors of that subspace), so for vector fields there are covector fields also called one-forms, ω

example $\mathcal{N}$ = integral submanifold
$= \{x \in M \text{ s.t. } h(x) = c\}$

the differential of h (also corresponding to the gradient of h)

$$dh = \left(\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n}\right)$$ is a one-form

this comes from differentiating a function (here $h(x)=c$) i.e. a constraint on the manif.)

By construction $L_f h = 0 = dh(f) = \sum \frac{\partial h}{\partial x} f = 0$

as we have already seen. //

there are however other one-form (i.e. covectors) which are not differentials of functions

→ non-integrable constraints

→ nonholonomic constraints

they represent

1. - kinematic constraints (forbidded directions of the velocity)
   - for instance: bodies in contact which roll without slipping
   - example: rolling wheels, wheeled vehicles object manipulation with fingers

2. - dynamical constraints: conservation of angular momentum for mechanical systems

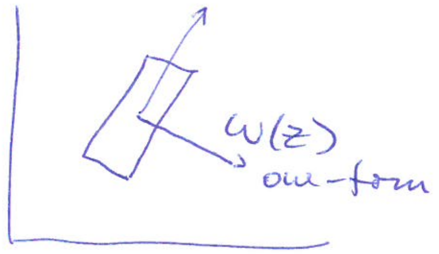
- example falling cat, spacecrafts ~~with a~~ with momentum wheels (rotors)

ex rolling wheels

we know that $\Delta = \text{span}\{g_1, g_2\}$ non integrab.

$\Rightarrow$ there must be a nonholonomic constr.

one-form = constraint among velocities



$$\omega(z)\dot{z} = \omega(z)\sum_{i=1}^{2} g_i(z)\,u_i$$

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix}\begin{bmatrix} v\cos\theta \\ v\sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$= v\sin\theta\cos\theta - v\sin\theta\cos\theta = 0 \quad //$$

If $\Delta = \text{span}\{g_1, g_2\}$ distribution

$\Omega = \text{span}\{\omega\}$ <u>co-distribution</u>

$\Omega = \Delta^{\perp}$ co-distribution annihilates the distribution

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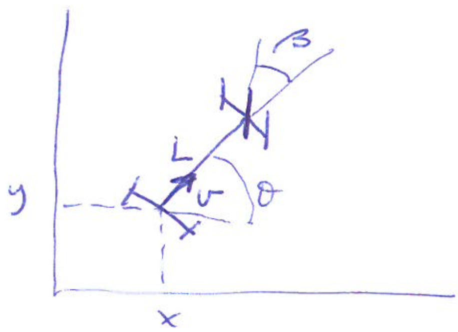
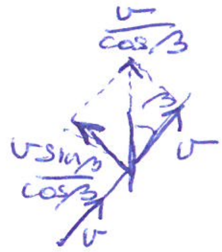
Δ non-integrable $\Leftrightarrow$ Ω non-integrable

Δ completely non integrab $\Leftrightarrow$ Ω completely nonholonom

<u>ex</u> <u>car-like vehicle</u>

$\mathcal{M} = \mathbb{R}^2 \times S^1 \times S^1$





$$z = \begin{bmatrix} x \\ y \\ \theta \\ \beta \end{bmatrix}$$

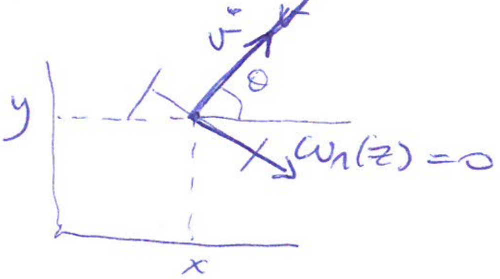
drift-less rank-2 distr.

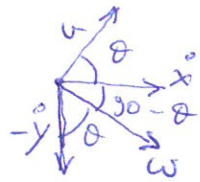
$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \frac{v}{L}\tan\beta \\ \dot{\beta} = \omega \end{cases}$$

$$\dot{z} = g_1(z)v + g_2(z)w = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\beta}{L} \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w$$

$\Delta = \mathrm{span}\{g_1, g_2\}$ has rank 2 (except at singular points)

intuitively:
2 nonholonomic constr:
1) back wheels





$\dot{x}\cos(90-\theta)$
$= \dot{x}\sin\theta$
$-\dot{y}\cos\theta$

$$\Rightarrow \omega_1(z) = \dot{x}\cos(90-\theta) - \dot{y}\cos\theta = 0$$
$$= \dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
$$= v\cos\theta\sin\theta - v\sin\theta\cos\theta = 0$$

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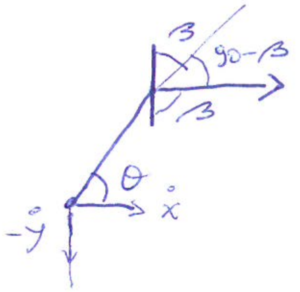
2) front wheels

$\dot{x}\sin(\theta+\beta)$

$-\dot{y}\cos(\theta+\beta)$

$-L\cos\beta\,\dot{\theta}=0$



$\omega_2(z)=0$

$$\omega_2=\dot{x}\sin(\theta+\beta)-\dot{y}\cos(\theta+\beta)-L\cos\beta\,\dot{\theta}=$$

$$=\dot{x}\left(\sin\theta\cos\beta+\cos\theta\sin\beta\right)-\dot{y}\left(\cos\theta\cos\beta+\sin\theta\sin\beta\right)-L\cos\beta\,\dot{\theta}=$$

$$=\left[\dot{x}(\sin\theta+\cos\theta\tan\beta)-\dot{y}(\cos\theta+\sin\theta\tan\beta)-L\dot{\theta}\right]\cos\beta$$

$$=\left[v\cancel{\cos\theta\sin\theta}+v\cos^2\theta\tan\beta-v\cancel{\cos\theta\sin\theta}+v\sin^2\theta\tan\beta-\cancel{L}\,v\frac{\tan\beta}{\cancel{L}}\right]\cos\beta$$

$$=\left[v(\cos^2\theta+\sin^2\theta)\tan\beta-v\tan\beta\right]\cos\beta=$$

$$=v(\tan\beta-\tan\beta)\cos\beta=0$$

ok!

compute the brackets

$$[g_1,g_2]=0-\frac{\partial g_1}{\partial z}g_2=\begin{bmatrix}0&0&-\sin\theta&0\\0&0&\cos\theta&0\\0&0&0&\frac{1}{L\cos^2\beta}\\0&0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\\\frac{-1}{L\cos^2\beta}\\0\end{bmatrix}=g_3(z)$$

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$$g_4 = [g_3, g_1] = \frac{\partial g_1}{\partial z} g_3 - \frac{\partial g_3}{\partial z} g_1 = \begin{bmatrix} \frac{\sin\theta}{L\cos^2\beta} \\ -\frac{\cos\theta}{L\cos^2\beta} \\ 0 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix}}_{=0}$$

at regular points ($\beta \neq \pm\pi/2$)

~~further~~ $\bar{\Delta} = \text{span}\{g_1, g_2, g_3, g_4\}$ has dim 4

$\Rightarrow \dim \bar{\Delta} = \dim T_x M = \dim M$

(at singular point $\beta = \pm\frac{\pi}{2}$ the car just turns around a circle $\Rightarrow$ cannot control it!)

at regular points we have STLC. //

Structure of nonholonomic (driftless) syst.

Given $\Delta = \text{span}\{g_1, \dots, g_m\}$ $m < n$ regular

when $\bar{\Delta} = T_p M$ the system is said <u>completely nonholonomic</u> (or completely non-integrable)

Filtration

from Δ, construct the sequence of distributions:

$\Delta_1 = \Delta$

$\Delta_2 = \Delta_1 + [\Delta_1, \Delta_1]$ i.e. v.f. + 1st. level Lie brackets

$\Delta_3 = \Delta_2 + [\Delta_1, \Delta_2]$

$\vdots$

$\Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}]$ v.f. + i-th level Lie br.

def the chain $\Delta_1, \Delta_2, \dots \Delta_i$ is called the filtration of Δ

def the filtration is regular in p_0 if
$$\mathrm{rank}(\Delta_i(p)) = \mathrm{rank}(\Delta_i(p_0)) \quad \forall p \in B(p_0) \quad \forall i$$

by construction: $\Delta_1 \subseteq \Delta_2 \subseteq \dots \subseteq \Delta_i$

If the filtration is regular, then $\exists\, k \in \mathbb{N}$ s.t. $\Delta_{k+1} = \Delta_k \Rightarrow$ filtration terminates and Δ_k is involutive $\Delta_k = \bar{\Delta}$

$\Rightarrow \Delta_{k+j} = \Delta_k \quad \forall j$

k = <u>degree of nonholonomy</u> of Δ

- If rank Δ_k ($=$ rank $\bar{\Delta}$) $= n$

  $\Rightarrow$ LARC holds $\Rightarrow$ Chow theorem

  $\Rightarrow$ STLC holds

- If rank $\Delta_k < n$ $\Rightarrow$ Frobenius theorem

  $\Rightarrow$ Δ_k integrable

  $\Rightarrow$ control system lives on a leaf of the foliation, i.e. on $\mathcal{N}_{c_i} \subset \mathcal{M}$ for some c_i, with $\dim(\mathcal{N}_{c_i}) =$
  $= \text{rank}(\Delta_k)$

For a regular filtration, denote $r_i \triangleq \text{rank}(\Delta_i)$, $i = 1, \dots k$

<u>def</u> the <u>growth vector</u> of Δ is $r \triangleq \{r_1, \dots r_k\}$

<u>Prop</u> the growth vector is <u>invariant to feedback equivalence</u>

meaning: given $\dot{x} = \sum_{i=1}^{m} g_i(x) u_i \quad (*)$

use
- change of state variable
  $z = \Phi(x)$ diffeomorphism
- change of input
  $v = \Psi(x, u)$ diffeomorphism in u

$$\begin{bmatrix} z \\ v \end{bmatrix} = \chi(x, u) = \begin{bmatrix} \Phi(x) \\ \Psi(x, u) \end{bmatrix}$$

then the system

$$\dot{z} = \left(\frac{\partial \Phi}{\partial x} \sum g_i(x) u_i \right) \Bigg|_{\substack{x = \Phi^{-1}(z) \\ u = \Psi^{-1}(\Phi^{-1}(z), v)}}$$

has the same growth vector as $(*)$

Consequence: only systems with the same growth vector r can be mapped into each other by means of feedback equiv.

<u>examples</u> of growth vectors for rank-2 distr.
- rolling wheel $r = \{2, 3\}$
- kinematic car $r = \{2, 3, 4\}$
- n-trailer system $r = \{2, 3, 4, 5, \ldots n\}$