(n87)

PART II: CONTROLLABILITY

Today: <u>Basics of differential geometry</u>

nonlinear control system $\dot{x} = f(x,u)$

- autonomous
- $x \in M$ smooth (i.e. C^∞) or real analytic (i.e. admits a power series expansion) manifold
- $u \in \mathcal{U}$ class of admissible control funct.
  - * bounded measurable
  - * piecewise const.
  - * smooth
- $f : M \times \mathcal{U} \to TM$ is an (input-parameter) vector field

$\mathcal{V}(M)$ = set of all vector fields over M

Murray - Li - Sastry "A Mathematical Introduction to Robotic Manipulation"

Nijmeijer - van der Schaft "Nonlinear Dynamical Control Systems"

A. Isidori "Nonlinear Control Systems"

<u>Manifold</u> is a subset of $\mathbb{R}^n$ defined by some smooth hypersurface, defined implicitely (like $\eta_i(x)=0$ from last time) or explicitely (like $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$ $x_2 = h(x_1)$ from last time)

ex $M = \{x \in \mathbb{R}^n \text{ s.t. } \eta_i(x) = 0 \quad i=1,\dots, n-m\}$

If rank of the Jacobian

$$\operatorname{rank} \begin{bmatrix} \frac{\partial \eta_1}{\partial x_1} & \cdots & \frac{\partial \eta_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial \eta_{n-m}}{\partial x_1} & \cdots & \frac{\partial \eta_{n-m}}{\partial x_n} \end{bmatrix} = m$$

then by the implicit function theorem M is a smooth manifold of dim m

<u>example</u>: Sphere $S^{n-1} = \{x \in \mathbb{R}^n \text{ s.t. } \|x\|_2^2 = 1 = \sum_{i=1}^n x_i^2\}$
$n-1$ dimensional manifold

example: Ball in $\mathbb{R}^n$ $B^n = \{x \in \mathbb{R}^n \text{ s.t. } \|x\|_2^2 \le 1\}$
special kind of manifold:
<u>manifold with boundary</u> $\dim(\operatorname{int}(B^n)) = n$
$\dim(\partial(B^n)) = n-1$

<u>example</u> (Lie) group of matrices

$$GL(n) = \{ A \in \mathbb{R}^{n \times n} \text{ s.t. } \det(A) \neq 0 \}$$

= general linear group

group + manifold = Lie group

<u>def</u>: <u>group</u> G: set of elements with an operation (multiplication)

$$G \times G \longrightarrow G$$

endowed with

1) identity element I
(i.e. $A \cdot I = A$)
2) inverse element : $A \in G \Rightarrow \exists A^{-1} \in G$
(i.e. $\forall A \in G \; \exists B \in G$ s.t. $AB = I$)

<u>example</u> (simplest example of Manifold)

$$M = \mathbb{R}^n$$

<u>example</u> of manifold: any open set of $\mathbb{R}^n$

geometric definition of manifold

<u>def</u> a <u>manifold</u> is a topological space with an atlas of charts of local coordinates

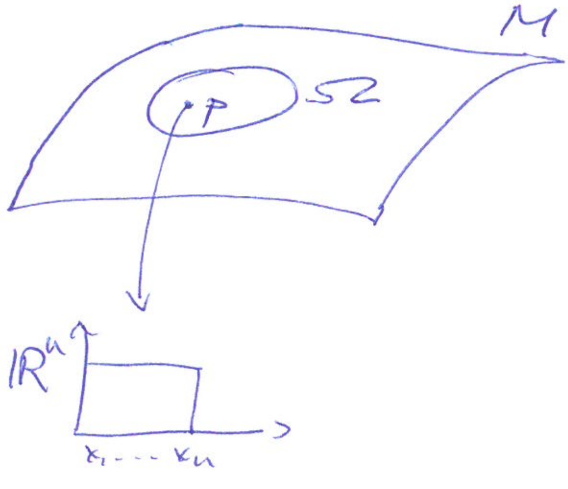
topological space : a sets with a notion of topology i.e. whose subsets are open subsets s.t.

- union of open sets is open
- intersection of open sets is open
- S and $\emptyset$ are open sets

a manifold of dim n locally looks like $\mathbb{R}^n$ and local charts make this identification precise

For any point $p \in M$
$\exists$ open set $\Omega \subset M$
which can be mapped
via ~~a~~ local chart ϕ
into ~~the~~ an open set of $\mathbb{R}^n$

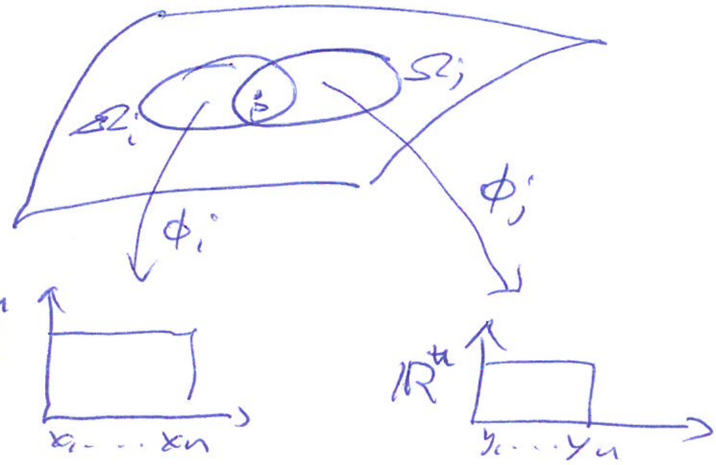
$x = \phi(p)$



We have always implicitly assumed that our statespace locally looks like $\mathbb{R}^n$ (an open domain of $\mathbb{R}^n$, more precisely)

Each point of M can have a different chart $\Rightarrow$ (Ω_i, ϕ_i) form an atlas of charts s.t. they are compatible on the overlaps

$x = \phi^\#(p)$ = local coord. description of p



$\Rightarrow$ ϕ has to be invertible (a bijection)

$y = \phi_j^\#(p)$

$x = \phi_i^\#(p)$ $\Rightarrow$ $y = \phi_j^\#(\phi_i^{-1}(x))$ also bijection

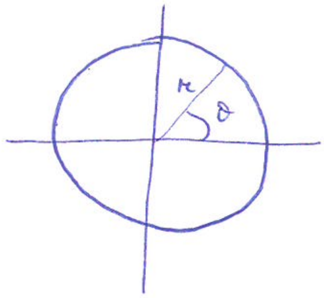
$p = \phi_i^{-1}(x) = \phi_j^{-1}(y)$

- charts (Ω_i, ϕ_i) have to be smooth
- charts (Ω_i, ϕ_i) have to be countably many
- atlas : $\bigcup_i \Omega_i = M$

<u>example</u> circle in $\mathbb{R}^2$ $S^1 \subset \mathbb{R}^2$

$S^1 = \{x_1^2 + x_2^2 = 1\}$

polar coordinates



r, θ

for instance $r = 1$, $-\pi \le \theta \le \pi$

S^1 has dim 1 ⇒ only 1 coordinate

$\phi : S^1 \to \mathbb{R}$

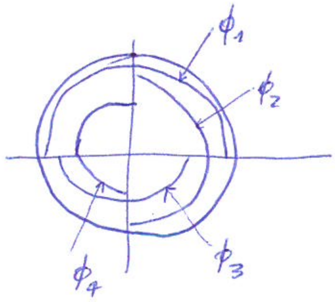
$x_1 = \cos\theta$ $\qquad -\pi \le \theta \le \pi$

then $x_2 = \sqrt{1 - x_1^2}$

However $\phi(\theta) = \cos\theta$ is not invertible everywhere

($\theta = \pm\pi/2$)

⇒ must choose multiple charts

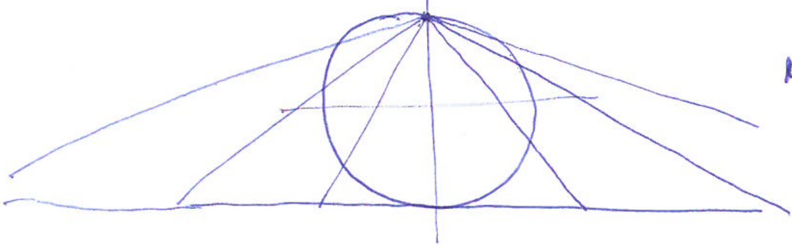


4 charts $x_1 = \phi_1 = \cos\theta$ $\quad x_2 = \sqrt{1 - x_1^2}$

$x_2 = \phi_2 = \sin\theta$ $\quad x_1 = \sqrt{1 - x_2^2}$

⋮

construction of charts is not unique



in projective geometry one would map S^1 to $\mathbb{R}$ in this way

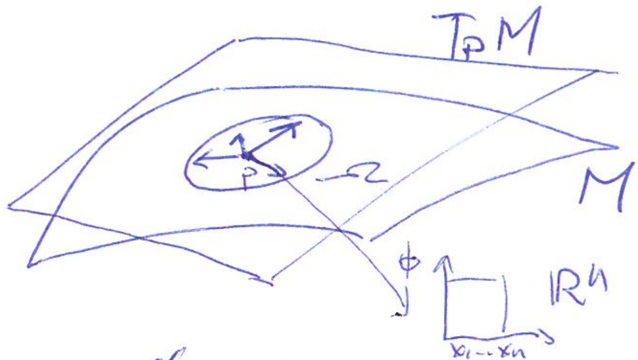
why <u>invertible charts?</u> because calculations are done in the $\mathbb{R}^n$ space, not in M: if $\psi : M \to \mathbb{R}$ is a function then we use $\psi \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ to do everything charts are used to define a differentiable structure on the manifold

<u>def</u> a function $\psi : M \to \mathbb{R}$ is <u>differentiable</u> if $\forall$ charts (Ω, ϕ) $\psi \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable

$\to$ <u>differentiable manifold</u>

<u>Tangent space</u> at $p \in M$

is the collection of all tangent vectors at p
= collection of all derivations at p



T_pM is a vector space

If $(x_1 \ldots x_n) \in \mathbb{R}^n$ is the set of local coordinates induced by the chart

$\Rightarrow \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ set of local coord. induced on T_pM by the chart.

$\dim(T_pM) = \dim(TM)$

<u>tangent vector</u> = element of T_pM

$v \in T_pM$ has a coordinate description

$$v = v_1 \frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n}$$

or simply $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ if there is no ambiguity

<u>Tangent bundle</u> = collection of all tangent spaces at all points

$$TM \triangleq \bigcup_{p \in M} T_pM$$

TM has dim $2n$ and elements (p, T_pM)

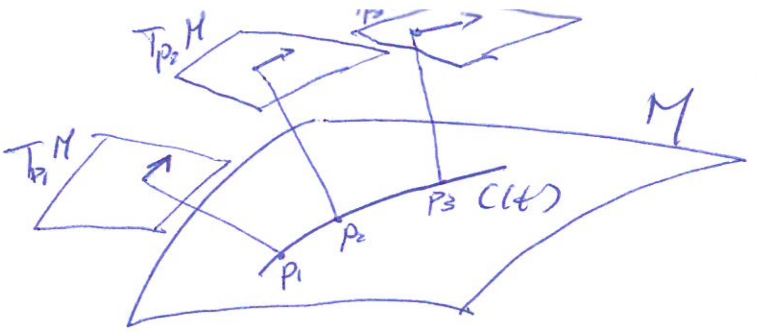
<u>Vector fields</u> = smooth map $M \to TM$ that we use to represent odes on manif.

$$f : M \to TM$$
$$p \mapsto f(p)$$

in coordinates $x_1 \dots x_n$

$$f(p) = f_1(p) \frac{\partial}{\partial x_1} + \dots + f_n(p) \frac{\partial}{\partial x_n} \quad \text{(operator form)}$$

when there is no ambig: $x \equiv p$ $f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$



time interv. $I \in \mathbb{R}$

consider a curve $c(t)$ on the manifold

$$c: I \longrightarrow M$$
$$t \longmapsto c(t)$$

since T_pM represents all derivatives, if we have $\dfrac{d\,c(t)}{dt} = f(c(t)) = f \circ c(t)$

for some v. f. $f \in \mathcal{V}(M)$, then $f(c(t))$ "collects" the tangent vectors to $c(t)$ at all points and $c(t)$ can be considered an <u>integral curve</u> of the vector field f. (also called <u>flow</u>)

<u>example</u>

if $M = \mathbb{R}^n$ local chart $(x_1, \dots x_n)$ is also global

$\Rightarrow$ a single chart suffice for M

$\Rightarrow \left(\dfrac{\partial}{\partial x_1} \cdots \dfrac{\partial}{\partial x_n}\right)$ global chart also for TM

since $TM = \mathbb{R}^n \times \mathbb{R}^n$

canonical coordinates: $x_i \Leftrightarrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}$

induce $\dfrac{\partial}{\partial x_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}$

$\Rightarrow$ I can write the vector field

$$f(x) = f_1(x)\frac{\partial}{\partial x_1} + \dots + f_n(x)\frac{\partial}{\partial x_n} \quad \text{as}$$

$$f(x) = f_1(x)\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} + \dots + f_n(x)\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}$$

ex linear vector field on $M = \mathbb{R}^n$

$$f(x) = Ax \qquad A \in \mathbb{R}^{n\times n} \qquad x \in \mathbb{R}^n$$

$$Ax = \begin{bmatrix}a_{11} & \dots & a_{1n}\\ \vdots & & \\ a_{n1} & \dots & a_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \sum_j a_{1j}x_j\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} + \dots + \sum_j a_{nj}x_j\begin{bmatrix}0\\ \vdots\\ \vdots\\1\end{bmatrix}$$

$$f_i(x) = \sum_j a_{ij}x_j$$

Notation for flow (or integral curve) along f

$$\Phi^f(t,p) = \Phi^f_t(p)$$

$$\Phi^f_t : \underset{\subset\mathbb{R}}{I} \times M \to M$$

$$(t,p) \mapsto \Phi^f_t(p)$$

f is called an infinitesimal generator of the flow Φ^f_t

$$f(p) = \frac{d}{dt}\Phi^f_t(p)\Big|_{t=0}$$

def a vector field is said complete if its integral curve is defined for $I = \mathbb{R}$ (i.e. also for "negative times")

flow composition

$$\Phi^f_{t_2} \circ \Phi^f_{t_1}(p) = \Phi^f_{t_1+t_2}(p)$$

- If f is complete flow composition form a group
- If f not complete (e.g. $t \geq 0$) then flow compos. form a semigroup (all properties of a group except inverse) –
  It is called a one-parameter semigroup (param is time)

If f complete: for any $\Phi^f_t(p)$ $\exists$ an inverse $\Phi^f_{-t}(p)$ s.t. $\Phi^f_{-t} \circ \Phi^f_t(p) = p$

<u>ex</u> linear system (linear v.f.) $f(x) = Ax$

$$x(t_1) = \Phi^f_{t_1}(x_0) = e^{At}x_0$$

$$x(t_2) = \Phi^f_{t_2} \circ \Phi^f_{t_1}(x_0) = e^{At_2}e^{At_1}x_0 = e^{A(t_1+t_2)}x_0$$

$$= \Phi^f_{t_1+t_2}(x_0)$$

If $t_2 = -t_1 \Rightarrow e^{A(t_2+t_1)}x_0 = e^{A(t_1-t_1)}x_0 = x_0$

Operations with vector fields

Given: $f \in \mathcal{V}(M)$ i.e.

- vector field $f : M \to TM$

$$p \mapsto (p, f(p)) = \left(p, \sum f_i(p) \frac{\partial}{\partial x_i}\right)$$

$$\left(\text{or even } = \left(\phi, \sum f_i(\phi^{-1}(x)) \frac{\partial}{\partial x_i}\right)\right)$$

- real-valued function, smooth

$$\psi : M \to \mathbb{R} \qquad \psi \in C^\infty$$

1) Lie derivative of a function w.r.t. a v.f.

= directional derivative of a function along the v.f.

$$L_f \psi : M \to \mathbb{R}$$
$$p \mapsto L_f \psi(p)$$

in coordinates:

$$L_f \psi(p) = \left(\sum_i f_i(p) \frac{\partial}{\partial x_i}\right) \psi(p)$$
$$= \sum_i f_i(p) \frac{\partial \psi(p)}{\partial x_i} = \underbrace{\frac{\partial \psi(p)}{\partial x}}_{\text{gradient of } \psi} f(p)$$

same as the directional derivative we saw for Lyapunov function

Multiple Lie derivative can be applied:

$f, g \in \mathcal{V}(M)$ $\quad \psi : M \to \mathbb{R}$ function

since $L_f \psi : M \to \mathbb{R}$ is also a function

$$L_g L_f \psi \triangleq L_g (L_f \psi) = \frac{\partial (L_f \psi(p))}{\partial x} g(p)$$

Applying k-times the same v.f. ⇒ recursive def

$$L_f^k(\psi) \triangleq L_f(L_f^{k-1}\psi) \quad \text{with } L_f^0 \psi = \psi$$

2) Lie bracket of two v.f. $[\cdot, \cdot]$

given $f, g \in \mathcal{V}(M)$, and a test function $\psi : M \to \mathbb{R}$, the Lie bracket is the bilinear map

$$[\cdot, \cdot] : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$$
$$(f, g) \longmapsto [f, g]$$

computed as (along test function)

$$[f, g]\psi(p) = L_f L_g \psi(p) - L_g L_f \psi(p)$$

in local coordinates $(x_1 \dots x_n)$ (no need of test funct.)

$$[f,g](x) = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobians, computed at $x = \phi(p)$

<u>proof</u> (that the two expressions are the same)

From $f = \sum_i f_i \frac{\partial}{\partial x_i}$ $\quad g = \sum_i g_i \frac{\partial}{\partial x_i}$ $\quad L_f \varphi = \sum_i f_i \frac{\partial \varphi}{\partial x_i}$

$$L_g L_f \varphi = \sum_j g_j \frac{\partial}{\partial x_j}\left(\sum_i f_i \frac{\partial \varphi}{\partial x_i}\right)$$

$$= \sum_j \sum_i g_j \left(\frac{\partial f_i}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + f_i \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)$$

hence

$$L_f L_g \varphi - L_g L_f \varphi = \sum_j \sum_i \left(f_j\left(\frac{\partial g_i}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + \cancel{g_i \frac{\partial^2 \varphi}{\partial x_i \partial x_j}}\right) - g_j\left(\frac{\partial f_i}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + \cancel{f_i \frac{\partial^2 \varphi}{\partial x_i \partial x_j}}\right)\right)$$

$$= \sum_i \sum_j \left(f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j}\right)\frac{\partial \varphi}{\partial x_i} \qquad \text{sum over } j$$

$$= \sum_i \left(\frac{\partial g_i}{\partial x} f - \frac{\partial f_i}{\partial x} g\right)\frac{\partial \varphi}{\partial x_i}$$

which is the coord. exp. for a v.f. $h = \sum h_i \frac{\partial}{\partial x_i}$

$$= \left(\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g\right)\varphi(x)$$

//

consequence: you do not need a test function to compute $[\cdot,\cdot]$, only the v.f. f and g.

Alternative notation: "ad"

$$ad_f g(p) \triangleq [f,g](p)$$

Properties of Lie bracket

- $[f,f]=0$ a v.f. commutes with itself
- recursive calculation of v.f.

$$[f,[f,g]] = ad_f^2 g$$

$$ad_f^k g = [f, ad_f^{k-1} g], \quad ad_f^0 g = g$$

- two vector fields commute if $[f,g]=0$

- $[L_f, L_g] = L_f L_g - L_g L_f \overset{!}{=} L_{[f,g]}$

<u>example</u> linear v.-f. $f(x)=Ax$, $g(x)=Bx$

$$[Ax,Bx]=[f,g](x)=\frac{\partial g}{\partial x}f-\frac{\partial f}{\partial x}g=\frac{\partial(Bx)}{\partial x}Ax-\frac{\partial(Ax)}{\partial x}Bx$$

$$=BAx-ABx=(BA-AB)x=[B,A]x$$

$\Rightarrow$ matrix commutator. $=-[A,B]x$

<u>example</u> $f(x)=a=\text{const}$, $g(x)=Bx$ linear

$$[f,g](x)=Ba=\text{const}.$$

<u>example</u> $f(x)=a=\text{const}$ $g(x)=b=\text{const}$.

$$[f,g](x)=0$$

Lie algebra

<u>def</u> A Lie algebra is a vector space V (over the real field $\mathbb{R}$) endowed with a bilinear operation (the Lie bracket) s.t.

1) (bilinearity) $[\alpha_1 f_1 + \alpha_2 f_2, g] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g] \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$
$\forall f_1, f_2, g \in V$
2) (skew-commutativity): $[f, g] = -[g, f]$
3) (Jacobi identity)
$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$
$\forall f, g, h \in V$

<u>def</u> a Lie subalgebra is a vector subspace $W \subset V$ s.t. $[f, g] \in W \quad \forall f, g \in W$
(invar. w.r.t. the Lie bracket operation)

<u>example</u> of subalgebra of $gl(3)$

$so(3) \overset{!}{=}$ special orthogonal lie algebra
= matrix lie algebra of skew-symmetric matrices

$= \{ A \in gl(3) \text{ s.t. } A^T = -A \}$

$\dim(so(3)) = 3$ (lie algebra is a vector space hence it has a basis of equal dim)

basis of $so(3)$:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

commutation relations

$$[A_1, A_2] = A_3$$

$$[A_2, A_3] = A_1$$

$$[A_3 A_1] = A_2$$

Any element of $so(3)$ has the form $A = \sum \alpha_i A_i = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$

meaning of basis elements: infinitesimal rotations around 3 axes of rotation