Existence of a CQLF is a sufficient but not necessary condition for uniform asympt. stab. of a switched system –

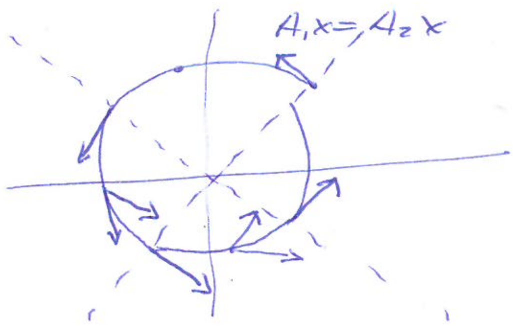
example $A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}$ $P = \{1, 2\}$

the two systems do not share any QLF (show it!)

However the switching system $\dot{x} = A_\sigma x$ is globally unif asympt. st. for all σ switch. signals

To see it: look at worst-case switching

The two systems $A_1 x$ and $A_2 x$ are collinear on the dashed lines



In between one of the two points more outside than the other (and they switch roles on the dashed lines

$\Rightarrow$ worst-case consists of following always the one pointing most outwards

Even doing that the trajectory produced converge to the origin –

Another way to show unif.-as-stab: use higher order homogeneous polynomials in place of quadratic polynomials

ex $V(x) = x^T P x = x^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} x = [x_1 \; x_2] \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$= p_1 x_1^2 + 2 p_2 x_1 x_2 + p_3 x_2^2$ homogeneous polyn. of order 2

If I can find higher order homogeneous polyn.

$$V(x) = \sum_j p_j x_1^{k_{j1}} x_2^{k_{j2}} \quad \text{s.t. } k_{j1} + k_{j2} = 4 \text{ (for instance)}$$

s.t. $V(x)$ is pos. def, then I can try to use it as Lyapunov function.

ex $V(x) = (x^T P x)^2$ 4th order homog. polyn, but not useful because it "replicates" twice the same Lyapunov funct.

A systematic way to compute higher order homog. polyn. funct. which are automatically pos. def.

Use <u>Kronecker products</u>. $\otimes$

For matrices $A, B \in \mathbb{R}^{n \times n}$ $\quad A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ a_{21} B & & & \\ \vdots & & & \\ a_{n1} B & & \cdots & a_{nn} B \end{bmatrix}$

For a vector $x \in \mathbb{R}^n$ $\quad \tilde{x} = x \otimes x = \begin{bmatrix} x_1 x \\ \vdots \\ x_n x \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ \vdots \\ x_1 x_n \\ \vdots \\ x_n x_1 \\ \vdots \\ x_n^2 \end{bmatrix}$ all monomials having power 2 $\in \mathbb{R}^{n^2}$

$A \otimes B \in \mathbb{R}^{n^2 \times n^2}$ $\quad x \otimes x \in \mathbb{R}^{n^2}$

$$\mathring{x} = Ax$$

For $\tilde{x}$ a linear dynamics becomes :

$$\frac{d}{dt}\tilde{x} = \frac{d}{dt}(x \otimes x) = \mathring{x} \otimes x + x \otimes \mathring{x} = Ax \otimes x + x \otimes Ax$$

$$= (A \otimes I)\tilde{x} + (I \otimes A)\tilde{x} = \underbrace{(A \otimes I + I \otimes A)}_{\substack{\text{Kronecker sum} \\ \triangleq A \oplus A}}\tilde{x}$$

$\Rightarrow$ ~~A~~ quartic function can be constructed in $\tilde{x}$

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x} \qquad \tilde{x} \in \mathbb{R}^{n^2} \quad P \in \mathbb{R}^{n^2 \times n^2}$$

$$\mathring{V}(\tilde{x}) = \mathring{\tilde{x}}^T P \tilde{x} + \tilde{x}^T P \mathring{\tilde{x}} = \tilde{x}(A \oplus A)^T P \tilde{x} + \tilde{x}^T P (A \oplus A)\tilde{x}$$

$$\overset{!}{=} \tilde{x}((A \oplus A)^T P + P(A \oplus A))\tilde{x}$$

If I call $\tilde{A} = A \oplus A$ I get a Lyapunov eq in $\mathbb{R}^{n^2 \times n^2}$

$$\tilde{A}^T P + P\tilde{A}$$

For a single linear system there is no advantage in using polynomials of order higher than 2 ($A^T P + PA < 0$ for $P > 0$ is an iff condition for stab. of A)

However ~~for~~ when searching for a CLF, a Homog. Polyn. Lyap. funct is less conservative than a Quadratic Lyap. funct.

$$\begin{cases} \tilde{A}_i^T P + P\tilde{A}_i < 0 \\ P > 0 \end{cases}$$ is a less conservative suff. cond.

For the example: an 8-th order homog. polyn. common Lyap fun exists (not a QCLF, not a 4HPCLF or 6-HPCLF)

Center Manifold theory (Ch 8, Khalil)

Recall that for an autonomous linear system

$\dot{x} = f(x) \qquad f(0)=0$

we can investigate stability of $\bar{x}=0$ by looking at the linearization $\dot{x} = Ax \quad A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}$

we had:

nonlinear syst.		linearization
$\bar{x}=0$ loc. asympt. st.		
$\Uparrow$	$\Leftarrow$	$\bar{x}=0$ asympt. st.
		$\Updownarrow$
$\bar{x}=0$ loc. exp. st.	$\Leftarrow$	$\bar{x}=0$ exp. st.

If $Re[\lambda_i] \leq 0$ and $Re[\lambda_i]=0$ for some i, then linearization is inconclusive

Center manifold theory studies these inconclusive cases. It does so by "projecting" the dynamics on the manifold which ~~is the~~ "prolong" the eigenspace of the λ_i with $Re[\lambda_i]=0$

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manifold (more details next time): it is
a k-dimensional smooth hypersurface in $\mathbb{R}^n$
represented by an implicit eq. like $\eta(x)=0$
where $\eta: \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$

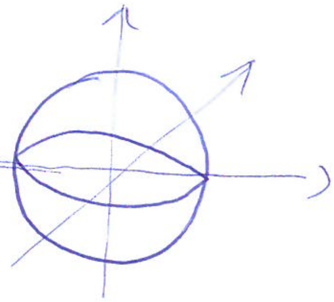
$$\mathcal{M} = \text{manifold} = \{ x \in \mathbb{R}^n \mid \eta(x) = 0 \}$$

example: sphere in $\mathbb{R}^n$:
$$\mathbb{S}^n = \{ x \in \mathbb{R}^n \mid \|x\|_2^2 = 1 \} = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1 \}$$

is a $(n-1)$-dimensional manifold

ex: $\mathbb{S}^2$ in $\mathbb{R}^3$

2-dim manf. in $\mathbb{R}^3$



manifold is invariant for $\dot{x} = f(x)$ if

$$\eta(x(0)) = 0 \Rightarrow \eta(x(t)) = 0 \quad \forall t \geq 0$$

Consider nonlinear autonomous system

$$\dot{x} = f(x) \qquad 0 = f(0) \qquad f \in C^2$$

Compute linearization at $\bar{x} = 0$

Jacob. $A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}$

rewrite system as

$$\dot{x} = Ax + \Big(f(x) - \underbrace{\frac{\partial f}{\partial x}(0)}_{A}\, x\Big) = Ax + \tilde{f}(x)$$

where by construction $\begin{cases} \tilde{f}(0) = 0 \\ \frac{\partial \tilde{f}}{\partial x}(0) = 0 \end{cases}$

Since we are interested in the undecidable case,

A has
- k eigenvalues with $Re[\lambda_i] = 0$
- n-k eigenval. with $Re[\lambda_i] < 0$

$\Rightarrow$ make a change of basis T s.t.

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{array}{l} \} \; Re[\lambda_i] = 0 \quad k\text{-dim} \\ \} \; Re[\lambda_i] < 0 \quad (n-k)\text{-dim} \end{array}$$

$$\Rightarrow \quad \begin{bmatrix} y \\ z \end{bmatrix} = Tx$$

system becomes:

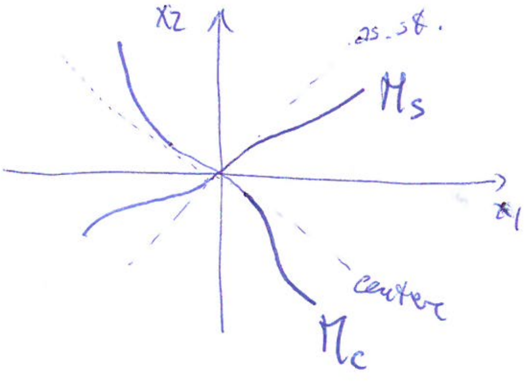
$$(*) \begin{cases} \dot{y} = A_1 y + g_1(y,z) \\ \dot{z} = A_2 z + g_2(y,z) \end{cases} \quad \text{with} \quad (**) \begin{cases} g_i(0,0) = 0 \\ \frac{\partial g_i}{\partial y}(0,0) = 0 \\ \frac{\partial g_i}{\partial z}(0,0) = 0 \end{cases}$$

y is the "center" part
z is the asympt. st. part.

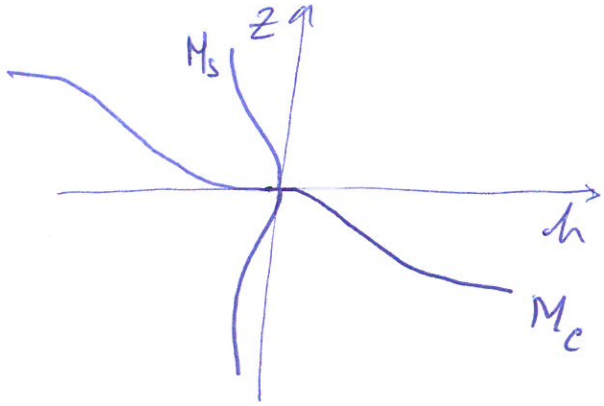
properties are valid by construction from $\tilde{f}(0) = 0 \quad \frac{\partial \tilde{f}}{\partial x}(0) = 0$

<u>def</u> An invariant manifold $z = h(y)$ $h \in C^\infty$ for the system $(*)$ is called a <u>center manifold</u> if $h(0) = 0$, $\frac{\partial h}{\partial y}(0) = 0$

Meaning: locally I split $\mathbb{R}^n$ into 2 vector subspaces, one for the eigenvalues of A_1 ($\to$ center), the other for the eigenvalues of A_2 ($\to$ as. st.) _ then I "prolong" these eigenvalues to manifolds



change basis



center manifold $M_c = \{(y,z) \text{ s.t. } z = h(y)\}$

M_c is tangent to $y=0$ plane at 0 (and passes through origin)

$$\Rightarrow \begin{cases} h(0)=0 \\ \frac{\partial h}{\partial y}(0)=0 \end{cases}$$ $\Rightarrow$ $h(y)$ has terms on y of power 2 or higher (by construction)

Meaning M_s = stable manifold, is less important since it contains as. stable directions $\Rightarrow$ what matters is what happens on the center manif

existence thm:

<u>thm</u> For the system (*) with $g_i \in C^2$ and satisfying (**) $\exists\, \delta > 0$ and $h \in C^\infty$ defined in $\|y\| < \delta$ s.t. $z = h(y)$ is a center manifold for the system.

Center manifold is invariant $\Rightarrow$

$(y(0), z(0)) \in M_c$ i.e. $z(0) = h(y(0))$ $\Rightarrow$

$z(t) = h(y(t))$ $\forall t \geq 0$ $\quad ((y(t), z(t)) \in M_c \ \forall t)$

$\Rightarrow$ we can replace $z(t)$ with $h(y(t))$ in the equation

⇒ <u>reduced system</u> (of dim k and living in M_c)

$$\dot{y} = A_1 y + g_1(y, h(y))$$

this reduced system decides as. stability of the entire system. → <u>reduction principle</u>

<u>thm</u> If $\bar{y}=0$ of the reduced system is asymptotically st. (unstable) then $\bar{x}=0$ of the original system is as. st. (unstable)

<u>Proof</u> idea: "outside M_c" asymptotic converg. rules (even exponential)

i.e. if on a ball B_r $z(0) \neq h(y(0))$ then $(y(t), z(t))$ is not in M_c –

Denote $w = z - h(y)$ the deviation from M_c

then $\|w(t)\| \leq e^{-\lambda t} \|w(0)\|$

⇒ M_c is locally attracting as a set

⇒ only what happens on M_c matters //

the deviation w helps in finding an expression for $\mathcal{M}_c$ -

Change basis : $\begin{bmatrix} y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ w \end{bmatrix}$ $\quad z = w + h(y)$

$$\dot{y} = A_1 y + g_1(y, w + h(y))$$

$$\dot{w} = \dot{z} - \frac{\partial h}{\partial y}\dot{y} = A_2(w + h(y)) + g_2(y, w + h(y)) - \frac{\partial h}{\partial y}\left(A_1 y + g_1(y, w + h(y))\right)$$

the center manifold $\mathcal{M}_c$ in the coord. $\begin{bmatrix} y \\ w \end{bmatrix}$ is $w=0$. Hence being in $\mathcal{M}_c$ and staying in $\mathcal{M}_c$ means $w(t) \equiv 0 \rightarrow \dot{w}(t) \equiv 0$

i.e.

$$0 = \Psi(h(y)) = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left(A_1 y + g_1(y, h(y))\right)$$

this is a PDE that must be solved in $h(y)$ in order to find the $\mathcal{M}_c$, with boundary conditions $\begin{cases} h(0) = 0 \\ \frac{\partial h}{\partial y}(0) = 0 \end{cases}$

→ difficult !

Very often an explicit expression for $h(y)$ cannot be found. However $h(y)$ can be approximated via a Taylor expansion.

<u>thm</u> If a function $\phi(y)$, with $\phi(0)=0$ $\frac{\partial \phi}{\partial y}(0)=0$, C^2 can be found such that

$$\Phi(\phi(y)) = O(\|y\|^p) \text{ (infinitesimal of order } p)$$

for some $p>1$, then for $\|y\|$ sufficiently small it is $h(y)-\phi(y)=O(\|y\|^p)$

and the reduced system can be represented as

$$\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^{p+1})$$

Asympt. stab (unstab.) can be deduced from it

$p>1$ because Taylor expansion of $h(y)$ starts at $p=2$ by construction

$$h(y) = k_1 y^2 + k_2 y^3 + \dots$$

or $h(y)=0$, also possible

<u>example</u> $\dot{x}_1 = x_1 x_2$
$\dot{x}_2 = -x_2 + a x_1^2$

$$A = \frac{\partial f}{\partial x}\Big|_0 = \begin{bmatrix} x_2 & x_1 \\ 2ax_1 & -1 \end{bmatrix}\Big|_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y = x_1 \\ z = x_2 \end{cases} \qquad \begin{cases} \dot{y} = 0 + yz \\ \dot{z} = -z + ay^2 \end{cases}$$

$M_c = \{ (y,z) \text{ s.t. } z = h(y) \}$ = center manifold

on M_c, reduced system is $\dot{y} = y\,h(y)$

Must find $\cancel{z} = h(y)$ that define center manifold

Use the PDE

$$\Psi(h(y)) = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left(A_1 y + g_1(y, h(y))\right) = 0$$

$$-h(y) + ay^2 - \frac{\partial h(y)}{\partial y}\left(y\,h(y)\right) = 0 \qquad \text{s.t. } h(0) = 0, \ \frac{\partial h}{\partial y}(0) = 0$$

this PDE is in general impossible to solve in $h(y)$

$\Rightarrow$ use Taylor ~~app~~ expansion as approximation, simplest approxim: $\phi(y) = 0$

$\Rightarrow ay^2 = 0$ Ok only if $a = 0$, not in general
syst is $\dot{y} = 0$ i.e. $\bar{y} = 0$ stable $\Rightarrow \bar{x} = 0$ stable but not asympt. st.

next simplest approximation: power 2 in h

$\phi(y) = k_2 y^2 \cancel{+ \mathcal{O}(|y|^3)}$ k_2 unknown coeff. $h(y) = \phi(y) + \mathcal{O}(|y|^3)$

plug in into the PDE:

$$\Psi(\phi(y)) = -k_2 y^2 + ay^2 - 2k_2 y (y k_2 y^2) = 0$$

If $k_2 = a \Rightarrow \Psi(\phi(y)) = -2k_2^2 y^4 = -2a^2 y^4 = \mathcal{O}(|y|^4)$
approximation

$\Rightarrow$ in the reduced system

$$\dot{y} = y\,h(y) = ay^3 + \mathcal{O}(|y|^4)$$

- when $a > 0$ reduced system is unstable
$\Rightarrow$ orig syst. is unstable

- when $a < 0$ reduced syst is as. stab
$\Rightarrow$ orig syst is as. st.

- when $a = 0$ ~~can use~~ PDE is $-h(y) - \frac{\partial h(y)}{\partial y}(y\,h(y)) = 0$
which is solved by $h(y) = 0$

$\Rightarrow$ reduced syst. is $\dot{y} = 0$ which is stable
(but not asympt. st.)

$\Rightarrow$ original syst is stable

Other concepts of stability

absolute, robust, practical, diagonal, semiglobal, ...

Systems given with inputs and outputs

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

1) Input-state stability

For linear systems: BIBO (Bounded Input Bounded Output) stability

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{free evol}} + \underbrace{C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced evol.}}$$

BIBO stab:

$$\|u\| < k_u \Rightarrow \|y\| < k_y$$

asympt. stab. of A $\Rightarrow$ BIBO stability

(for minimal realiz: $\Leftarrow$)

to show it:

$$\|y(t)\| \leq \underbrace{\|C\| e^{-\lambda t} \|x(0)\|}_{\text{expon. stab.}} + \underbrace{\|C\| \frac{K}{\lambda} \|B\| \sup \|u\|}_{\text{boundedness}}$$

If nonlin syst. with $u=0$ is expon. st. then the same type of input-to-state stability behavior can be expected also for the nonlin system

However you need to have also a global Lipschitz condition, which is difficult to get in practice

<u>example</u> $\dot{x} = -x + (x^2+1)u$

If $u=0$ then $x=0$ is exp. st. for $\dot{x} = -x$
However if $u=1$ then $\dot{x} = -x + x^2 + 1$ diverges to ∞
$\Rightarrow$ unstable

$\Rightarrow$ "small signals" stability: for $\|x(0)\| < r_1$, $\|u(t)\| < r_2$

$$\|y\|_\infty \leq c_1 \|u\|_\infty + c_2 \|x(0)\|$$

2) Dissipative / passive systems

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

associated with a supply rate function

$$w : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$$

def system is dissipative w.r.t. the supply rate $w(u,y)$ if $\exists$ a positive def. storage function $S : \mathcal{X} \to \mathbb{R}_+$ s.t. the following dissipation inequality holds:

$$S(x(t)) \leq S(x(0)) + \int_0^t w(u(\tau), y(\tau))\, d\tau$$

or, differentiating

$$\dot{S}(x) \leq w(u,y)$$

Meaning

- supply rate = "infinitesimal energy" supplied to the system (power flow into the syst.)
- storage function = energy stored in the syst.

- dissipation : stored energy is bounded by the total energy supplied externally
  i.e. dissipative system cannot internally create energy.
- Storage function is related to Lyapunov function

<u>Thm</u> In a dissipative system If $w(0,y) \leq 0 \ \forall y$ and $\bar{x}=0$ is a minimum of S, then $\bar{x}=0$ is a locally stable eq. of $\dot{x}=f(x,0)$ and $V(x)=S(x)-S(0) \geq 0$ is a Lyapunov funct

<u>Proof</u> $\dot{V}(x)=\dot{S}(x) \leq w(0,y) \leq 0$ when $u=0$

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2') <u>Passive systems</u> are a subclass of dissipative systems - They have supply rate function

$$\omega(u, y) = u^T y \qquad \text{(implicitly: n° of inputs = n° of outp.)}$$

<u>def</u> the system $\dot{x} = f(x, u)$
$y = h(x, u)$ is

i) <u>passive</u> if it is dissipative w.r.t. the supply rate $w(u, y) = u^T y$
i.e. $u^T y \geq \dot{S}(x)$

ii) <u>lossless</u> if $u^T y = \dot{S}(x)$

iii) <u>strictly passive</u> if $u^T y \geq \dot{S}(x) + \psi(x)$
with $\psi(x)$ pos. def.

Meaning of passivity: energy that flows into the system ($u^T y$) is more than the increase of stored energy ($\dot{S}$)

example: in a circuit with a resistor the power that flows in ~~goes to~~ "changes" the capacitors and inductors, but gets lost in the resistors n86

example integrator $\begin{cases} \dot{x} = u \\ y = x \end{cases}$ $G(s) = \frac{1}{s}$

supply rate $w(u,y) = uy$

Storage funct. $S(x) = \frac{1}{2}x^2 > 0$

$\Rightarrow \dot{S}(x) = \frac{2}{2}x\dot{x} = uy$

$\Rightarrow$ lossless system

example low-pass filter $\begin{cases} \dot{x} = -x + u \\ y = x \end{cases}$ $G(s) = \frac{1}{s+1}$

supply rate $w(u,y) = uy$

Storage funct $S(x) = \frac{1}{2}x^2$

$\Rightarrow \dot{S}(x) = x\dot{x} = -x^2 + uy$

$\Rightarrow uy > \dot{S}(x) \Rightarrow$ strictly passive

$\Rightarrow \bar{x} = 0$ is globally as. st. when $u = 0$

thm system is passive, $S(x) > 0$ $\Rightarrow \bar{x} = 0$ is stab. when $u = 0$

system is strictly passive, $S(x) > 0$ $\Rightarrow \bar{x} = 0$ is asympt. st. when $u = 0$