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LaSalle invariance principle formalizes this "weaker" Lyapunov condition.

Consider $\dot{x} = f(x)$; $x(t)$ solution.
$f: \mathcal{D} \to \mathbb{R}^n$, Lipschitz cont.

<u>def</u> A point p is a (positive) <u>limit point</u> of $x(t)$ ~~for the system~~ if $\exists$ a sequence of times $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, s.t. $x(t_n) \to p$ as $n \to \infty$.

<u>def</u> a <u>limit set</u> of $x(t)$ is the set of all its limit points, denoted L^+.

- limit set can contain equilibria, limit cycles.

<u>def</u> A set M is a (positively) <u>invariant set</u> for the system if $x(0) \in M \Rightarrow x(t) \in M$ $\forall t \geq 0$

<u>Lemma</u> If a trajectory $x(t)$ bounded, $x(t) \in \mathcal{D}$ $\forall t \geq 0$ then L^+ is a nonempty, compact, invariant set. Moreover $x(t)$ approaches L^+ as $t \to \infty$.

here "approaches": $\mathrm{dist}(x(t), L^+) = \inf_{y \in L^+} \|x(t) - y\| \xrightarrow{t \to \infty} 0$

thm (LaSalle invariance principle)

Let $\Omega \subset \mathcal{D}$ be a compact invariant set for the system. Let $V: \mathcal{D} \to \mathbb{R}$ be a C^1 function s.t. $\mathring{V}(x) \leq 0$ in Ω. Let $E \subset \Omega$ be the set of points in which $\mathring{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Proof

Ω compact, invar. $\Rightarrow L^+ \subset \Omega$

$V(x)$ decreasing, cont $\Rightarrow$ lower bounded

$\Rightarrow$ it is $\lim_{n \to \infty} V(x(t_n)) = a$ for any $\{x(t_n)\} \xrightarrow{n \to \infty} p \in L^+$

On L^+: $\mathring{V}(x) = 0$ $\left(\text{since } V(x(t_n)) \xrightarrow[x(t_n) \to p]{t_n \to \infty} \text{const}\right)$ since L^+ is an invariant set

relationship of sets: $L^+ \subseteq M \subset E \subset \Omega$ //

$$\underset{\substack{\text{invar}\\ \mathring{V}(x)=0}}{L^+} \subseteq M \subset \underset{\mathring{V}(x)=0}{E} \subset \underset{\substack{\text{compact}\\ \text{invar}}}{\Omega}$$

Notice that $V(x)$ need not be pos. def ($V(x) > 0$ can be used still to constr. Ω compact, invariant) example (1-dim adaptive control system)

$\dot{y} = ay + u$ $\qquad$ u = control input

task: render $y=0$ asympt. stable,
without knowing the parameter a

adaptive control law

$$u = -k\,y \quad , \quad \dot{k} = \gamma y^2 \qquad \gamma > 0$$

idea: increase the gain when error is large
decrease it when error is small
(error $\sim y^2$, γ fixed)

closed-loop system:

call $\left\{ \begin{array}{l} x_1 = y \\ x_2 = k \end{array} \right.$

$$\left\{ \begin{array}{l} \dot{x}_1 = -(x_2 - a)\, x_1 \\ \dot{x}_2 = \gamma x_1^2 \end{array} \right.$$

line $x_1 = 0$ is the set of equilibria

want to show that $(x_1, x_2) \xrightarrow{t \to \infty} (0, \bar{x}_2)$

Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2-b)^2 \quad \text{where } b > a$$

$$\dot V(x) = x_1\dot x_1 + \frac{1}{\gamma}(x_2-b)\dot x_2 =$$

$$= -x_1^2(x_2-a) + x_1^2(x_2-b) = -x_1^2(b-a) \le 0$$

$V(x)$ radially unbounded

Any set $\Omega_c = \{x \in \mathbb{R}^2 \mid V(x) \le c\}$ is a positively invariant bounded set

$E = \{x \in \Omega_c \mid x_1 = 0\}$ is an invariant set since any $x_1 = 0$ is an equil point

$\Rightarrow M = E$ in this case

$\Rightarrow$ (LaSalle invariance princ.) every traj approaches $M = E$ $\Rightarrow x_1 \to 0$

conclusion is global because of radial unboundedness

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Interesting case: when M contains only 0

<u>Corollary</u> (<u>Krasovskii – LaSalle</u>)

Let $\bar{x}=0$ be equil point for $\dot{x}=f(x)$, $0\in\mathcal{D}$

Let $V:\mathcal{D}\to\mathbb{R}$ be C^1 pos. def. and s.t.

$\dot{V}(x)\leq 0$ in $\mathcal{D}$. Let $S=\{x\in\mathcal{D} \text{ s.t. } \dot{V}(x)=0\}$

If no solution can stay identically in S (other than $x(t)\equiv 0$) then the origin is asymptotically stable.

Meaning: the largest invariant set (i.e. M) in S is just the origin – hence $L^+\subset M=\{0\}$

$\Rightarrow x(t)\to 0$. (S was called E in LaSalle principle)

- here we ask for V to be p.d.

- result can be made global if $\mathcal{D}=\mathbb{R}^n$ and V is also radially unbounded

<u>example</u> (similar to damped pendulum)

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - f(x_2)$

where $f(0)=0$ and $x_2 f(x_2) > 0$ if $x_2 \neq 0$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

then

$\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = \cancel{x_1 x_2} - \cancel{x_1 x_2} - x_2 f(x_2) \leq 0$

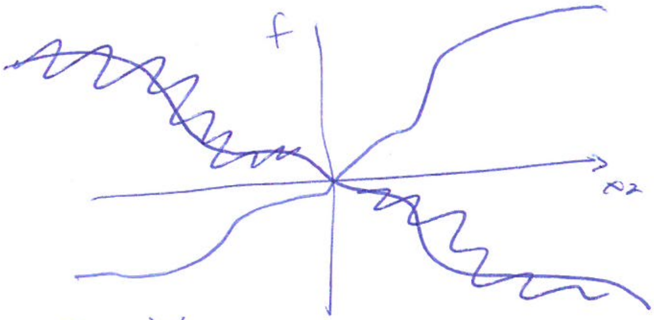
$\dot{V}(0) = 0$

$S = \{x \text{ s.t. } \dot{V}(x) = 0\} = \{x \text{ s.t. } x_2 = 0\}$

Is there any traj. other than $x \equiv 0$ in S?

$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$
$= -x_1$

general form of f:



these functions are called <u>passive</u> $x f(x) \geq 0 \ \forall x$

(1st and 3rd quadrant only)
(slope can be negative)

Lyapunov indirect method (aka: linearizat.)

Recall from linear systems

$$\dot{x} = Ax$$

$\lambda_1, \dots, \lambda_n$ = eigenval. of A
= sol. of char. eq. $\det(sI - A) = 0$

$\bar{x} = 0$ is

- asymptotically stable if $Re[\lambda_i] < 0 \quad \forall i$
- unstable if $Re[\lambda_i] > 0$ for some i
- stable (marginally stable) if $Re[\lambda_i] \leq 0$
  and for λ_i s.t. $Re[\lambda_i] = 0$ the
  Jordan blocks of λ_i have dim 1

(i.e. when algebr. multiplicity $q_i \geq 2$ it is
$\text{rank}[A - \lambda_i I] = n - q_i$)

(alg. multipl. of λ_i: multipl. of λ_i as zero of
$\det(\lambda I - A) = 0$)

reason if I have Jordan blocks with $\lambda_i = 0$

$$J = \begin{bmatrix} \ddots & & \\ & \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix} & \\ & & \ddots \end{bmatrix}$$

$\Rightarrow e^{Jt}$ contains terms like $te^{\lambda_i t} = te^0 = t$
which diverge as $t \to \infty$

Lyapunov eq: $V(x) = x^T P x$ $\quad P = P^T > 0$ (p.d.)

$$\Rightarrow \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x$$

$$= -x^T Q x \quad \text{with } Q = Q^T$$

where $-Q = PA + A^T P$

A has $Re[\lambda_i] < 0 \iff$ for any $Q = Q^T$ pos. def
$\exists$ a unique pos. def matrix P satisfying
the Lyapunov eq. $PA + A^T P = -Q$

For linear systems Lyapunov method is construct
and gives necessary and suff. cond. for asympt.
stability. It is easy to use: just take $Q = I$
and compute P.

for $\dot{x} = Ax$ $\quad \bar{x} = 0$ is <u>hyperbolic</u> equil point
if $Re[\lambda_i] \neq 0 \quad \forall i = 1, \ldots, n$

Back to nonlin. systems

$\dot{x} = f(x)$ $\quad f: \mathcal{D} \to \mathbb{R}^n$ Lipschitz cont.

$\bar{x} = 0$ equil p. $\quad f(0) = 0$

Compute series expansion around $\bar{x} = 0$

$$\dot{x} = f(x)\Big|_{x=0} + \frac{\partial f}{\partial x}\Big|_{x=0} (x - \bar{x}) + \underbrace{g(x)}_{\text{higher order terms}}$$

$$= 0 + Ax + g(x)$$

$$\frac{\|g(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0$$

<u>thm</u> (<u>Lyapunov indirect method</u>)

Consider $\dot{x} = f(x)$ and its linearization at $\bar{x} = 0$
$\dot{x} = Ax$

The equil point $\bar{x} = 0$ is

- locally asymptotically stable if $\mathrm{Re}[\lambda_i] = 0$ $\forall$ eigenval λ_i of A
- unstable if $\mathrm{Re}[\lambda_i] > 0$ for some eig λ_i of A
- undecidable from the linearization alone if $\mathrm{Re}[\lambda_i] \le 0$ $\forall \lambda_i$ of A and $\mathrm{Re}[\lambda_i] = 0$ for some λ_i of A

$\Rightarrow$ only for hyperbolic equil asympt stability can be decided through linearz.

Proof: $V(x) = x^T P x$

$$\dot V(x) = x^T P \left(Ax + g(x)\right) + \left(x^T A^T + g^T(x)\right) P x$$

$$\overset{!}{=} \underbrace{x^T \left(PA + A^T P\right) x}_{-x^T Q x} + \underbrace{2 x^T P g(x)}_{\text{negligible for } \|x\| \text{ small enough since } \frac{\|g(x)\|}{\|x\|} \to 0}$$

$\Rightarrow$ linearz. part rules the behav.

to show undecidability case:

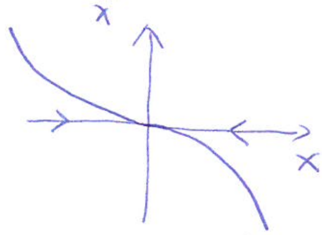
example: $\dot x = a x^3$

Jacobian: $A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left. 3 a x^2 \right|_{x=0} = 0$

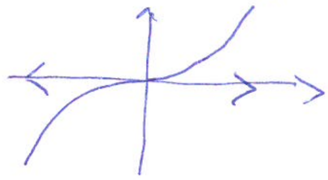
$\Rightarrow$ linearization cannot decide for the nonlin. s.

In fact

- if $a < 0$ $\quad \bar x = 0$ is asympt. stab.



- if $a > 0$ $\quad \bar x = 0$ is unst.
- if $a = 0$ $\quad \dot x = 0$ lin syst which is stable (trivial: $x(t) = \text{const}$)



$\Rightarrow$ all behaviors are possible for the nonlin system example: pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2 \qquad a, b > 0$$

equil $\bar{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\bar{x}_1 = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -a \cos x_1 & -b \end{bmatrix}$$

$$A_0 = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

λ_{12}: $\det(sI - A_0) = 0 \qquad \det \begin{bmatrix} s & -1 \\ a & s+b \end{bmatrix} = $

$$= s(s+b) + a = s^2 + bs + a = 0$$

$$\lambda_{12} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a} = -\frac{b}{2} \pm \frac{1}{2} \underbrace{\sqrt{b^2 - 4a}}_{< b^2} < 0$$

$\Rightarrow \bar{x} = 0$ is locally asympt. stable

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_1} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

λ_{12}: $\det(sI - A_1) = \det \begin{bmatrix} s & -1 \\ -a & s+b \end{bmatrix} = s(s+b) - a = 0$

$$\lambda_{12} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + 4a} \Rightarrow \begin{matrix} \lambda_1 > 0 \\ \lambda_2 < 0 \end{matrix} \Rightarrow \text{unstable}$$

NON-AUTONOMOUS SYSTEMS

Consider systems in which the right-hand side depends explicitely on time

$$\dot{x} = f(t, x)$$

⇒ solution is not a function of $t - t_0$ but of both t and t_0

⇒ all stability concepts that we have defined may depend on t

for example: stability

$\bar{x} = 0$ is stable if for each $\epsilon > 0$ ∃ $\delta(\epsilon, t_0) > 0$ s.t. $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \ \forall t \geq t_0$

⇒ not practical to have $\delta = \delta(\epsilon, t_0)$!

. to avoid this: uniform stability