Thm (Lyapunov direct method)

$\mathcal{D}$ open, maybe "small"

$\dot{x} = f(x)$, Lipschitz in $\mathcal{D}$, $0 \in \mathcal{D}$, $f(0) = 0$

Let $V: \mathcal{D} \to \mathbb{R}$ be C^1 function s.t.

- $V(0) = 0$, $V(x) > 0$ in $\mathcal{D} \setminus \{0\}$
- $\dot{V}(x) \leq 0$ in $\mathcal{D}$

then $\bar{x} = 0$ is stable.

Moreover, if $\dot{V}(x) < 0$ in $\mathcal{D} \setminus \{0\}$ then $\bar{x} = 0$ is asymptotically stable

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Proof combine two arguments

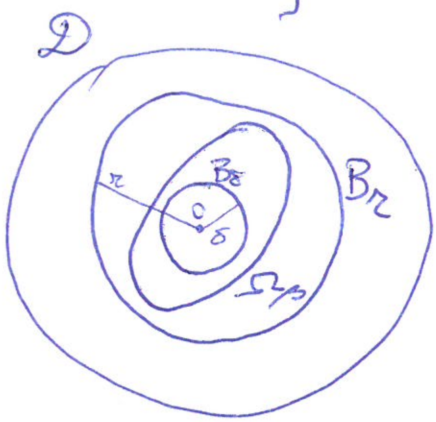
1) take a set $B_r = \{x \in \mathcal{D} \text{ s.t. } \|x\| \leq r\}$ s.t. in B_r: $\dot{V}(x) \leq 0$ $\Rightarrow$ if we have that $x(t)$ stays in B_r then $V(x(t)) \leq V(x(0))$

since $$V(x(t)) - V(x(0)) = \int_0^t \underbrace{\frac{d}{ds} V(x(s))}_{\leq 0} ds = \int_0^t \underbrace{\frac{\partial V}{\partial x} f(x(s))}_{\leq 0} ds$$

2) find a subset in B_r which is invariant for the system (so that you stay on it $\forall t$)

call $\alpha = \max_{\|x\|=r} V(x)$, take $\beta < \alpha$ $\beta \in (0, \alpha)$

take set $\Omega_\beta = \{ x \in B_r \text{ s.t. } V(x) \leq \beta \}$



from $V(x(t)) \leq V(x(0)) \leq \beta$
it follows that

$x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \ \forall t \geq 0$

since for any traj in Ω_β
$\dot{V}(x) \leq 0$ hence $V(x(t)) \leq \beta \ \forall t$ (and Ω_β is defined according to the level surfaces of $V(x)$)

(later on: we will see that sets like Ω_β are called invariant sets)

Ω_β compact (and invariant) $\Rightarrow$ solution $\exists$ unique $\forall t \geq 0$

In particular, $\exists \ \delta > 0$ s.t. $x(0) \in B_\delta$ ($\Rightarrow$ $V(x(0)) \leq \beta$) $\Rightarrow V(x(t)) \leq \beta \ \forall t$

$\Rightarrow x(t) \in \Omega_\beta$

$\Rightarrow x(t) \in B_r$ (since $\Omega_\beta \subset B_r$)

$\Rightarrow$ stability def holds (just call $\epsilon = r$)

- when $\mathring{V}(x) < 0$, to show asymptotic stability

  use that $V(x(t)) \leq V(x(0))$

- Must show that $V(x(t)) \to 0$ as $t \to \infty$

- By contradiction, assume $V(x(t)) \to c > 0$

- Find level surface of c and the ~~ball~~ set inside it

$$\Omega_c = \{ x \in B_r \text{ s.t. } V(x) \leq c \}$$

- Find ball strictly inside it : B_d

- Then $x(t)$ cannot enter inside B_d ($V(x)$ strictly decreasing but it stops at $c > d$)

- Denote

$$-\gamma = \max_{d \leq \|x\| \leq r} \mathring{V}(x) < 0$$

- Then $V(x(t)) = V(x(0)) + \int_0^t \mathring{V}(x(s))\,ds$

$$\leq V(x(0)) - \underbrace{\gamma t}_{\text{this term grows unbounded as } t \to \infty}$$

$\Rightarrow V(x(t))$ must become negative, which contradicts $c > 0$

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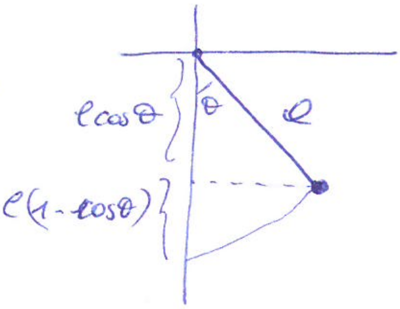
Obs: there is no systematic way to compute Lyapunov functions

<u>example</u> undamped pendulum

$$\dot{x}_1 = x_2 \qquad x_1 = \theta \;,\; x_2 = \dot{\theta}$$
$$\dot{x}_2 = -a \sin x_1 \qquad a = \frac{g}{l} \quad \frac{k}{m} = 0$$



- look at equil point $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- natural candidate Lyapunov function: energy

$E_{potential} + E_{kinetic}$ $\qquad E_{pot} = mgl(1-\cos\theta)$
(gravit. force . height)

$E_{kinet} = \frac{ml^2 x_2^2}{2} = [\text{mass} \cdot \text{vel}^2]$

$$\Rightarrow V(x) = \frac{E_{potent} + E_{kinetic}}{ml^2}$$
$$= \frac{mgl(1-\cos x_1)}{ml^2} + \frac{1}{2}\frac{ml^2 x^2}{ml^2} = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$V(x) > 0 \quad x \neq 0 \quad \text{in} \quad -2\pi < x_1 < 2\pi$

$V(0) = 0$

$\Rightarrow V(x)$ pos. def. funct.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = a \sin(x_1)\dot{x}_1 + x_2 \dot{x}_2$$
$$= a x_2 \sin x_1 - a x_2 \sin x_1 \equiv 0$$

i.e. tot. energy is conserved $V(x(t)) \equiv V(x(0))$

$\Rightarrow \dot{V}(x) \leq 0$ i.e. $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is stable but not asymptotically stable example pendulum with friction (n26)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2 \qquad \left(a = \frac{g}{\ell} \quad b = \frac{k}{m} \right)$$

same candidate Lyapunov function

$$V(x) = a(1-\cos x_1) + \frac{1}{2} x_2^2$$

gives:

$$\dot{V}(x) = a \dot{x}_1 \sin x_1 + x_2 \dot{x}_2$$
$$\stackrel{!}{=} a x_2 \sin x_1 - a x_2 \sin x_1 - b x_2^2 = -b x_2^2$$

$$\Rightarrow \dot{V}(x) \leq 0$$

again, Lyapunov direct methods predicts stability but not asymptotic stability (which intuitively we know should exist in this case...)

We will see below we can show asympt. stab. via the Krasowskii-LaSalle method.

homework: Show that $\exists \quad P = P^T > 0$

s.t. $V(x) = \frac{1}{2} x^T P x + a(1-\cos x_1)$ is a Lyapunov function s.t. $\begin{cases} V(x) > 0, & x \neq 0 \\ \dot{V}(x) < 0 & x \neq 0 \end{cases}$

$\Rightarrow$ asympt. stab. holds.

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0$ positive definite matrix

i.e. eigenv. of P are > 0

above: $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

n26bis

P pos. def $\Leftrightarrow$ $p_{11} > 0$, $p_{22} > 0$, $p_{11} p_{22} - p_{12}^2 > 0$

$$\dot{V}(x) = \frac{1}{2}\left(\dot{x}^T P x + x^T P \dot{x}\right) + a \sin(x_1)\, \dot{x}_1$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{bmatrix} + a x_2 \sin x_1$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} x_2 - p_{12} a \sin x_1 - b p_{12} x_2 \\ p_{12} x_2 - a p_{22} \sin x_1 - b p_{22} x_2 \end{bmatrix} + a x_2 \sin x_1$$

$$= \underline{p_{11} x_1 x_2} - a p_{12} x_1 \sin x_1 - \underline{b p_{12} x_1 x_2} + \underline{p_{12} x_2^2} - \underline{a p_{22} x_2 \sin x_1}$$
$$- b \underline{p_{22} x_2^2} + \underline{a x_2 \sin x_1}$$

$$= \underbrace{(p_{11} - b p_{12}) x_1 x_2} + \underbrace{a(1 - p_{22}) x_2 \sin x_1} + (p_{12} - b p_{22}) x_2^2$$
$$- a p_{12} x_1 \sin x_1$$

choose p_{ij} so as to cancel the mixed terms $x_1 x_2$, $x_2 \sin x_1$

$p_{22} = 1$, $p_{11} = b p_{12}$

$\Rightarrow$ it must be $b p_{12} \cdot 1 - p_{12}^2 > 0$ i.e. $p_{12}(b - p_{12}) > 0$

i.e. $p_{12} > 0$, $p_{12} < b$ $\Rightarrow$ $0 < p_{12} < b$. For instance $p_{12} = \frac{b}{2}$

$$\dot{V}(x) = \left(\frac{b}{2} - b\right) x_2^2 - \frac{ab}{2} x_1 \sin x_1 = -\frac{b}{2} x_2^2 - \frac{ab}{2} x_1 \sin x_1$$

< 0 for $x \in \mathcal{D} = \{ x \in \mathbb{R}^2 \text{ s.t. } |x_1| < \pi \}$

$\Rightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is asympt. stable!

message: Lyapunov conditions are <u>sufficient</u> but not necessary!

systematic methods for construction of Lyapunov functions are difficult.

<u>example</u>: variable gradient method

- idea: we seek for a vector $g(x)$ which is the gradient of a pos. def. funct. $V(x)$

$$g(x) = \nabla V(x) = \left(\frac{\partial V}{\partial x}\right)^T$$

and such that along the traj. of $\dot{x} = f(x)$ one has

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x) < 0$$

- $g(x)$ is gradient of a function $\Leftrightarrow$ Jacobian matrix $\frac{\partial g}{\partial x}$ is symmetric $\left(\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}\right)$

- steps:
  - choose $g(x)$ gradient of a function s.t. $g^T(x) f(x) < 0$
  - compute $V(x) = \int_0^x g^T(y)\, dy = \int_0^x \sum_{i=1}^n g_i(x)\, dy_i$
  - check if $V(x) > 0$

$g_i = \frac{\partial V}{\partial x_i} \rightarrow \frac{\partial g_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_i \partial x_j}$ Hessian

Global stability via Lyapunov direct method

- Conditions so far are local.
- If $\mathcal{D}=\mathbb{R}^n$ it can happen that $\bar{x}=0$ is globally asymptotically stable i.e.

$$\forall x(0)\in\mathbb{R}^n \quad \lim_{t\to\infty} x(t) = 0$$

- clearly $\bar{x}=0$ must be the only equil point (necess. cond.)

<u>def</u> $V:\mathbb{R}^n\to\mathbb{R}$ is <u>radially unbounded</u> if $V(x)\to\infty$ as $\|x\|\to\infty$

example: $V(x)=x_1^2+x_2^2$ pdf, radially unb in $\mathbb{R}^2$

$V(x)=x_1^2+\sin^2(x_2)$ locally pdf, but not radially unbounded

<u>thm</u> Consider $\dot{x}=f(x)$ $f:\mathbb{R}^n\to\mathbb{R}^n$ Lipschitz, $f(0)=0$

Let $V:\mathbb{R}^n\to\mathbb{R}$ be C^1 function s.t.

- $V(x)>0$ $\forall x\neq 0$, $V(0)=0$ (pd)
- $V(x)\to\infty$ as $\|x\|\to\infty$ (radially unb.)
- $\dot{V}(x)<0$ $\forall x\neq 0$

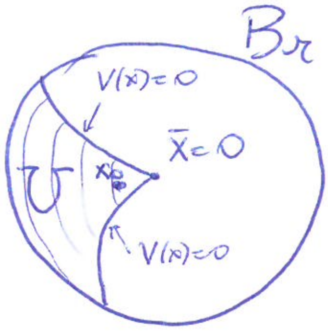
then $\bar{x}=0$ is globally asymptotically stable

Idea of proof: all level sets $V(x)\leq c$ are bounded

Instability theorems

<u>thm</u> (Cetaev) Consider $\dot{x} = f(x)$ Lipschitz in $\mathcal{D}$ $f(0)=0$. Let $V: \mathcal{D} \to \mathbb{R}$ be C^1 s.t.
$V(0)=0$ and $V(x_0)>0$ for some x_0 arbitrarily close to $\bar{x}=0$ (i.e. $\|x_0\|$ arbitr. small).
Let $U = \{x \in B_r \text{ s.t. } V(x)>0\}$
If $\dot{V}(x)>0$ in U then $\bar{x}=0$ is unstable

Idea of proof:



boundary of U: ∂B_r or the set of points in which $V(x)=0$

$\dot{V}(x)>0$ in $U \Rightarrow V(x)$ positive and growing $\Rightarrow$ can leave U only through ∂B_r
(a contradictory argument shows that $x(t)$ must leave U)
Since x_0 is arbitrarily small, the def. of stability is violated (it could be a saddle point, but that is also unstable ...)

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<u>example</u>: $\dot{x}_1 = x_1 + g_1(x)$
$\dot{x}_2 = -x_2 + g_2(x)$

where $g_1(\cdot), g_2(\cdot)$ locally Lipschitz s.t.

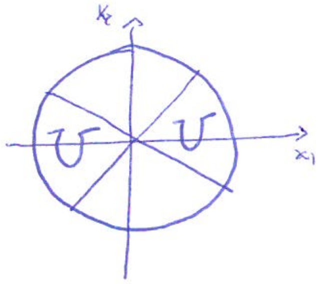
$$|g_1(x)| \leq k \|x\|_2^2 \qquad |g_2(x)| \leq k \|x\|_2^2$$

in $B(0, r)$. Consequently $g_i(0) = 0$

consider the function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$

on the line $x_2 = 0$, it is $V(x)\big|_{x_2=0} > 0$

also arbitrarily close to $\bar{x} = 0$



$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

'extra' terms can be bounded (in magnitude) by

$$|x_1 g_1(x) - x_2 g_2(x)| \leq \sum_{i=1}^{2} |x_i| \, |g_i(x)| \leq 2k \|x\|_2^3$$

$$\Rightarrow \dot{V}(x) \geq \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2 (1 - 2k\|x\|_2)$$

for B_r sufficiently small 1 dominates $2k\|x\|_2$

$\Rightarrow \dot{V}(x) > 0$ in B_r

$\Rightarrow$ cetaev thm is applicable

$\Rightarrow \bar{x} = 0$ is unstable

~~| this can be seen more easily by looking at the linearization~~ NONSENSE!

example (instability, easier)

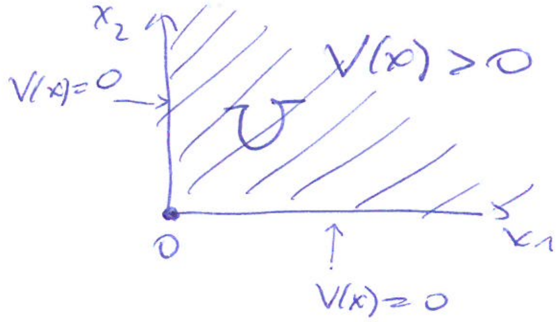
$$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = x_1^4 \end{cases} \qquad \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ equil point}$$

Consider the function $V(x) = x_1 x_2$

$V(0) = 0$

$V(x) > 0$ on $\mathbb{R}^2_+ = \{ x \in \mathbb{R}^2 \text{ s.t. } x_1 > 0, x_2 > 0 \} = \mathcal{U}$

$V(x) = 0$ in $\partial \mathbb{R}^2_+ = \{ x \in \mathbb{R}^2 \text{ s.t. } x_1 = 0 \text{ or } x_2 = 0 \} = \partial U$



in U it is:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = x_2 \dot{x}_1 + x_1 \dot{x}_2 = x_2^4 + x_1^5 > 0 \text{ in } U$$

$\Rightarrow$ Cetaev thm applies $\Rightarrow$ $\bar{x} = 0$ is unstable.

LaSalle invariance principle

Recall the example of pendulum with friction

example $\dot{x}_1 = x_2$ $(x_1 = \theta, \ x_2 = \dot{\theta})$

$\dot{x}_2 = -a \sin x_1 - b x_2$

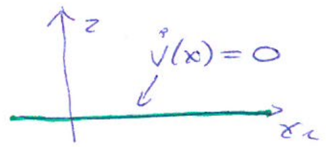
with the candidate Lyapunov function

$$V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2$$

it has:

$$\dot{V}(x) = -b x_2^2 \leq 0$$

i.e. only in $x_2 = 0$ it is $\dot{V}(x) = 0$
while in $x_2 \neq 0$ it is $\dot{V}(x) < 0$



Recall that, although we do not have explicitly the solutions of the system, $\dot{V}(x)$ is computed along its trajectories:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x)$$

Is there a trajectory of the system with $x_2(0) = 0$ and such that $x_2(t) = 0 \ \forall t$?

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow \sin x_1 = 0 \Rightarrow x_1 \equiv 0$$

$\Rightarrow$ only in $x_1 = 0$, $x_2 = 0$ this is possible!

$\Rightarrow$ even though Lyapunov direct method cannot be used, in practice on all trajectories of the system (other than $\bar{x} = 0$) it is $\dot{V}(x) < 0$