Department of Electrical Engineering (ISY)

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# Nonlinear Control Systems

## PhD course, Spring 2017

### Lecturer and examinator:

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### Aim:

The course aims at giving an overview of the main control problems and of some of the mathematical tools required in the analysis and synthesis of nonlinear control systems.

### Organization

12 lectures (2h each, once a week).

**Schedule**: Tuesdays at 13:15 in Algoritmen (B house, A-corridor). First class: March 7th.

### Prerequisites:

Knowledge in Linear Algebra, Automatic Control, Linear System Theory (basic or advanced) is assumed.

### Exam:

Hand-in exercises during the course, plus final take home exam. **Credits:** 6+3 p

### **Topics** (tentative):

- Lyapunov stability
  - stability by linearization
  - Lyapunov theorems
  - La Salle invariance principle;
- Nonlinear controllability
  - Differential geometric tools: manifolds, vector fields, Lie backets, Frobenius Theorem;
  - controllablity and Chow Theorem;
  - drift versus driftless systems, accessibility versus controllability;
- Geometric methods for control synthesis
  - system inversion and differential flatness
  - feedback linearization;
- Feedback stabilization
  - Brockett necessary condition
  - control Lyapunov functions

### Course material:

There is no single book covering all material that will be treated in the course. Some parts can be found in the following:

- H. Khalil. *Nonlinear Systems*, 3rd ed. Prentice Hall. 2002. (stability and stabilization)
- Murray-Li-Sastry. *A mathematical Introduction to Robotic Manipulation* CRC press 1994. (Controllability; nonholonomic systems; Lie algebras)
- F. Ticozzi. *Nonlinear Systems and Control*, Lecture Notes, 2016.

Page responsible: **Claudio Altafini**
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# Linear vs. Nonlinear

| | L | N |
|---|---|---|
| frequency analysis | yes | no |
| modal analysis | yes | no |
| superposition principle | yes | no |
| multiple isolated equil | no | yes |
| finite escape time | no | yes |
| non uniqueness of solution | no | yes |
| limit cycle (isolated periodic orbits) | no | yes |
| bifurcations | no | yes |
| chaos | no | yes |

# Multiple isolated equilibria

In a linear system $\dot{x} = Ax$

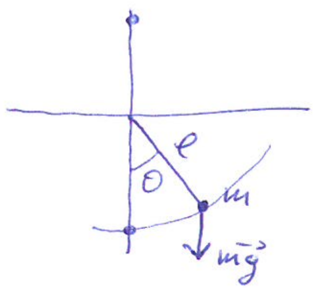
- $\bar{x}=0$ is always an equil point : $\dot{x} = 0 = A0$
- if $\bar{x} \neq 0$ is an equil point, then span$\{\bar{x}\}$ is a subspace of equilibria : $A\bar{x} = 0 \Leftrightarrow Ak\bar{x} = 0$ $k \in \mathbb{R}$

  $\to \bar{x} \in \ker(A) \to \det(A) = 0$
- if $\det(A) \neq 0$ then $\bar{x}=0$ is the only equil. point
- if $\det(A) = 0$ then $\exists$ subspace of equil points
- cannot have multiple isolated equil points

In a nonlin. system situation is different

example pendulum eq.

Newton law :




$$m l \ddot{\theta} + mg \sin\theta + \underbrace{k l \dot{\theta}}_{\text{friction term}} = 0$$

state space form : $x_1 = \theta$ , $x_2 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{cases}$$ nonlinear

equil : $x_2 = 0$, $\sin x_1 = 0$ $\to$ $\bar{x} = (0,0)$, $\bar{x} = (\pi, 0)$ isolated eq. points

# Finite escape time

For the linear system $\dot{x} = Ax$, if $A$ is stable (i.e. all eig of $A$ have $Re[\lambda_i] \leq 0$) then the trajectories are bounded. If $\exists \lambda_i$ s.t. $Re[\lambda_i] > 0$ then $\exists x_0$ s.t. the solution diverges

$$\lim_{t \to \infty} \| x(t) \| = \lim_{t \to \infty} \| e^{At} x(0) \| = +\infty$$

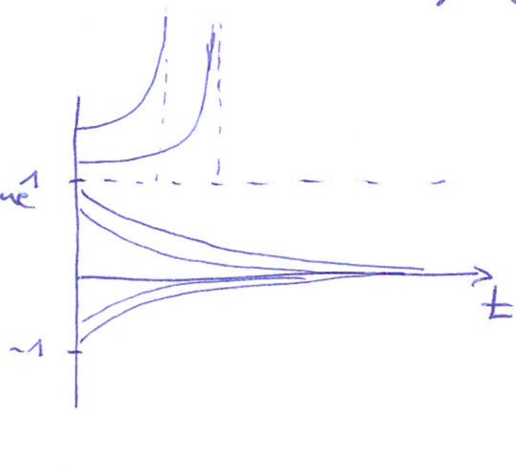
For each finite $t$, however, the solution is finite i.e $\|x(t)\| < +\infty \quad \forall t$ finite

For nonlinear systems, the solution may become $\infty$ even when $t$ is finite

<u>example</u>: $\dot{x} = -x + x^2$

$\bar{x} = 0$ equil point, if we look at the linearizx. around $0$, $\dot{\tilde{x}} = -\tilde{x}$, then the solution near $0$ is $\tilde{x}(t) = e^{-t}\tilde{x}(0)$, stable and converging to $0 = \bar{x}$.

But when $x_0 > 1$ the situation is different: divergence in finite time




Integrating the ODE

$$x(t) = \frac{e^{-t} x_0}{1 - x_0 + x_0 e^{-t}}$$

when $x_0 > 1$: denominator at $t=0$ is $1 - x_0 + x_0 = 1$ then it decreases until $0$ when $x_0 e^{-t} = x_0 - 1$
$\Rightarrow$ solution diverges to $+\infty$

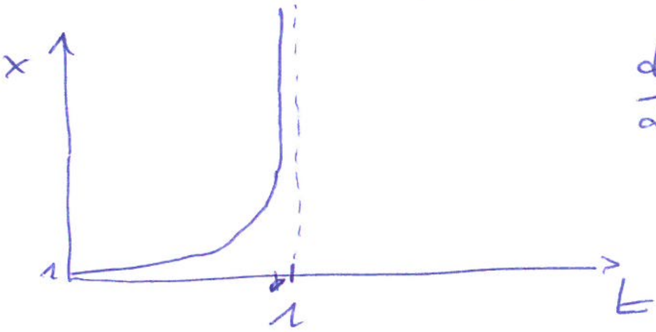
example (simpler)

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$

the solution is $x(t) = \frac{1}{1-t}$ (verify by differentiation)

$$\frac{dx}{dt} = -\frac{(-1)}{(1-t)^2} = \frac{1}{(1-t)^2} = x^2$$

solution




$x(t) \to \infty$ as $t \to 1$ $\Rightarrow$ finite escape time

$\Rightarrow$ solution exists only in $0 \leq t < 1$

more general solution

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$$

$$\frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x(t)} + \frac{1}{x_0} = t$$

- if $x_0 < 0 \Rightarrow x(t)$ exists in $[0, \infty)$
- if $x_0 > 0 \Rightarrow x(t)$ exists in $[0, \frac{1}{x_0})$

I can compute the maximal solution only because I could integrate explicitly the ODE. In general I cannot do it!

# Nonuniqueness of solutions

- linear systems : $\dot{x} = Ax$ has always a unique sol.
- nonlin. system need not have it !

**example** $\dot{x} = 2\sqrt{x}$

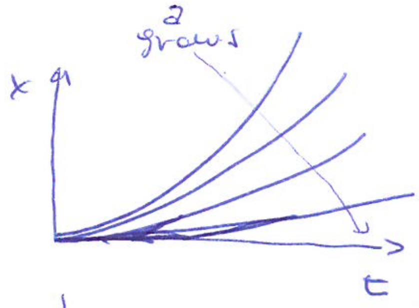
$x(0) = 0$

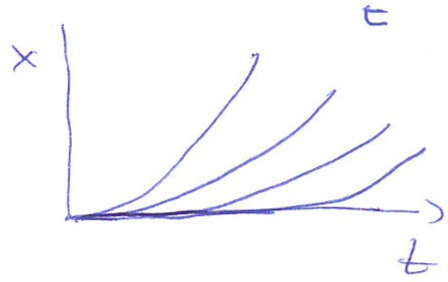
For each $a \geq 0$ the following is a solution

$$x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ (t-a)^2 & \text{if } t > a \end{cases}$$

In fact, differentiating :

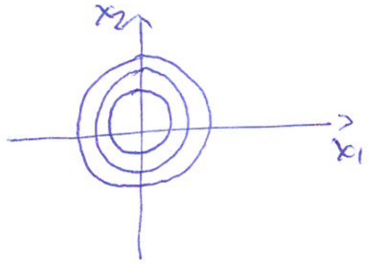
$$\frac{dx}{dt} = \begin{cases} 0 & 0 \leq t \leq a \\ \underbrace{2(t-a)}_{=2\sqrt{x}} & t > a \end{cases}$$



# (stable) limit cycle

Linear system $\dot{x} = Ax$ can have cyclic trajectories, but they can only be stable, never attractors of nearby trajectories (they are not isolated)




phase portrait

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

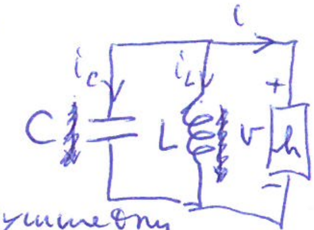
$\det(sI - A) = s^2 + \omega^2 = 0$

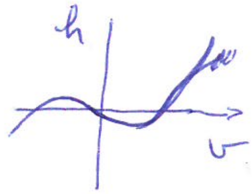
$\Rightarrow \lambda_{1,2} = \pm j\omega$

$$x(t) = e^{At} x(0) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} x(0)$$

- Nonlinear systems can have isolated periodic trajectories that attract nearby trajectories

<u>example</u> basic oscillator circuit




$i = h(v)$ nonlinear resistence of odd symmetry

$L, C > 0$

$h(0) = 0$

$\left.\frac{\partial h}{\partial v}\right|_0 = h'(0) < 0$

$C \frac{dv_C}{dt} = i_C$

$L \frac{di_L}{dt} = v_L$

Kirchoff law $\quad i_C + i_L + i = 0$

$$C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^{t} v(s)\,ds + h(v) = 0$$

differentiating w.r.t. time and multiply by $L$

$$LC\frac{d^2v}{dt^2} + v + L\,h'(v)\frac{dv}{dt} = 0$$

special case of Lienard's equations

- rescale time: $\tau = \frac{t}{\sqrt{CL}} \Rightarrow \frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt}$

$$\frac{d^2v}{d\tau^2} = CL\frac{d^2v}{dt^2}$$

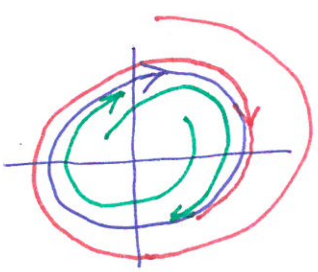
$$\Rightarrow \frac{d^2v}{d\tau^2} + \underbrace{\sqrt{\frac{L}{C}}}_{=\epsilon} h'(v)\frac{dv}{d\tau} + v = 0$$

Special case: <u>Van der Pol oscillator</u>: $h(v) = -v + \frac{1}{3}v^3$

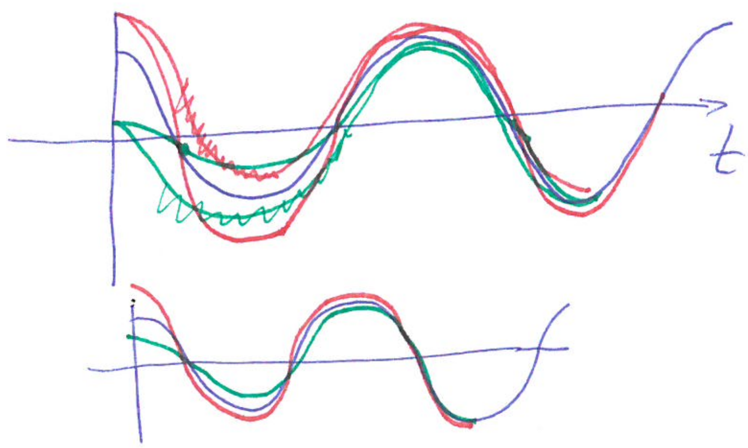
$$\ddot{v} - \epsilon(1-v^2)\dot{v} + v = 0 \qquad \epsilon = \sqrt{\frac{L}{C}}$$

state space model: $x_1 = v$, $x_2 = \dot{v}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - \epsilon h'(x_1)x_2 \end{cases}$$



stable limit cycle



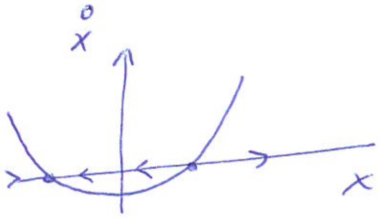
# Bifurcation

= qualitative change in solution of ODE as we change a parameter

- change in the number of equil. points
- change in the stability character of the equil points

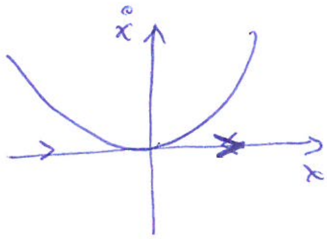
example: saddle-node bifurcation ("blue-sky" bifurc.)

$\dot{x} = r + x^2$



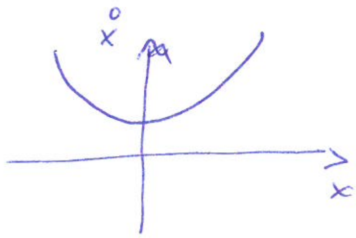

$r < 0$

2 equil
1 stable
1 unstable



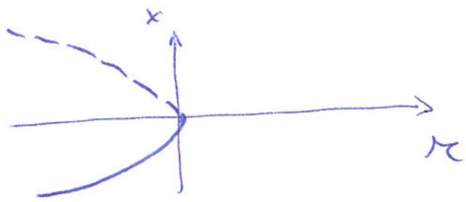

$r = 0$

2 equil
coalesce
half-stable




$r > 0$

no equil
point.




bifurcation diagram.

## Terminology

General form of a nonlinear system

$$(*) \qquad \dot{x} = f(t, x, u) \qquad t = \text{time} \in \mathbb{R}_+$$

$x$ = state vector $\in \mathcal{D} \subset \mathbb{R}^n$

$u$ = control input $\in \mathcal{U} \in \mathbb{R}^m$

$f : \mathbb{R}_+ \times \mathcal{D} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ vector field

System is autonomous (or time-invariant) if $t$ does not appear explicitly on the r.h.s

i.e. $\dot{x} = f(x, u)$

more precisely: $x(x_0, t, t_0) = x(t - t_0, x_0)$

examples:

- linear system

$$\dot{x} = Ax + Bu$$

- bilinear system

$$\dot{x} = Ax + Bxu$$

bilinear: linear in both $x$ and $u$

- control - affine

$$\dot{x} = \underbrace{f(x)}_{\text{drift}} + \sum_{i=1}^{m} g_i(x) u_i \qquad u_i \text{ scalar}$$

- driftless if $f(x) = 0$

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

- system without control

$$\dot{x} = f(x)$$

# Existence and uniqueness of solutions

initial value problem for ODE

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

- when $\exists$ a solution ?
- when is the solution unique ?

$f : \mathbb{R}_+ + \mathcal{D} \to \mathbb{R}^n$ we assume $\mathcal{D} \subset \mathbb{R}^n$ open

<u>def</u> <u>open</u> set if any $x_0 \in \mathcal{D}$ has a neighborhood inside $\mathcal{D}$

i.e. $\exists\ \epsilon > 0$ s.t. $\|x - x_0\| < \epsilon \Rightarrow x \in \mathcal{D}$

what norm ? any norm !

<u>def</u> norm $\|\cdot\|$ is an $\mathbb{R}$-valued function s.t.

1) $\cdot\ \|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$, $\|x\| = 0 \iff x = 0$

2) $\cdot\ \|x+y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in \mathbb{R}^n$ (triangle inequality)

3) $\|\alpha x\| = |\alpha|\,\|x\| \quad \forall\, \alpha \in \mathbb{R},\ x \in \mathbb{R}^n$

example : p-norm

$$\|x\|_p = \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p} \quad 1 \le p < \infty$$

$$\|x\|_\infty = \max_i |x_i|$$

$p=2$ : Euclidean norm

Prop all p-norms are equivalent : given $\alpha, \beta$ $\exists$ $c_1, c_2 \in \mathbb{R}_+$ s.t.

$$c_1 \|x\|_\alpha \le \|x\|_\beta \le c_2 \|x\|_\alpha \quad \forall x \in \mathbb{R}^n$$

Back to the initial value problem

- for existence of solution, continuity of $f(t,x)$ in both $t$ and $x$ is enough
  $$f \in C^0(\mathbb{R}_+ \times \mathcal{D})$$
- for existence and uniqueness : one can ask
  - continuity in $x$ of $f$
  - piecewise-continuity in $t$ of $f$ (so that to include for instance cases in which the control $u(t)$ is discontinuous ex : step response )

(Cauchy-Peano ; Picard-Lindelöf theorems ...)

- a local <u>Lipschitz condition</u>

$$\| f(t,x) - f(t,y) \| \leq L \| x-y \| \qquad L > 0$$

$\forall\, x, y \in B(x_0, r) = \{ x \in \mathbb{R}^n \text{ s.t. } \| x - x_0 \| < r \}$

$\forall\, t \in [t_0, t_1]$

<u>thm</u> the system $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ cont in $x$, piecewise cont in $t$ with the local Lipschitz condition has a unique solution over the interval $[t_0, t_0+\delta]$ for some $\delta > 0$

<u>Prop</u> • maximal interval in which the solution can be extended must be open $[t_0, t_0+\delta_{max})$, perhaps $\infty$ —

$L$ = Lipschitz constant (independent of $x, y, t$)

<u>Global Lipschitz condition</u>: when Lipschitz in $\mathbb{R}^n$

<u>meaning</u>: in $\mathbb{R}^1$, for $f: \mathbb{R} \to \mathbb{R}$ (autonomous)

the Lipschitz cond is $\dfrac{|f(y) - f(x)|}{|y-x|} \leq L$

$\Rightarrow$ "increment" in ODE is bounded even if $f$ is not differentiable

- differentiability is stronger than Lipschitz
- Lipschitz is stronger than continuous

**Lemma** Let $f(t,x) \in C^0([t_0,t_1] \times \mathcal{D})$.

Assume $\frac{\partial f}{\partial x}$ $\exists$ continuous on $[t_0,t_1] \times \mathcal{D}$

If in a convex $W \subset \mathcal{D}$ $\exists$ const $L > 0$ s.t.

$\| \frac{\partial f}{\partial x}(t,x) \| \leq L$ (norm of the Jacobian)

then $\| f(t,x) - f(t,y) \| \leq L \| x - y \|$ //

∘ meaning: $f$ has bounded partial derivatives

Here $\|A\|$ = matrix norm = induced p-norm

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$$

examples: Euclidean norm $\|A\|_2 = \left(\lambda_{max}(A^TA)\right)^{1/2}$ = largest singular value of $A$

$\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ row ~~column~~ sum

$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$ column ~~row~~ sum

**example** (of non uniqueness)

$\dot{x} = 2\sqrt{x}$ $\qquad \frac{\partial f}{\partial x} = \frac{2}{2} \frac{1}{\sqrt{x}}$ not bounded at $x = 0$

$\Rightarrow$ ODE is not locally Lipschitz at $x = 0$

$\Rightarrow$ multiple solutions exist

# Extension to global ~~uniqueness~~ existence and uniqueness

**thm** $f: \mathbb{R}_+ \times \mathbb{R}^n$ cont in $x$, piecewise cont in $t$ and satisfies a global Lipschitz cond

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \forall x,y \in \mathbb{R}^n \quad \forall t \in [t_0, t_1]$$

then the system $\dot{x} = f(t,x)$, $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$ // ($t_1$ can be arbitrarily large)

- global Lipschitz cond. is difficult to satisfy ⇒ conservative suff. cond.
- often you can have it only on compact sets

**def**: compact set = closed and bounded set

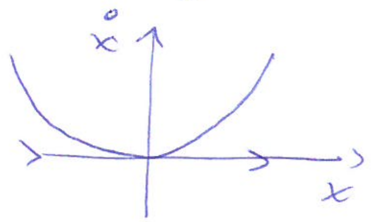
**def** closed set: if its complement is open.

- for autonomous systems: stronger suff. cond.

**thm** If $f \in C^1(\mathcal{D})$ i.e. cont with cont first derivative in $x$ and autonomous, then the solution exists and is unique $\forall x_0 \in$ compact and $\forall t \in [t_0, t_0 + \delta_{max})$

example (finite escape time)

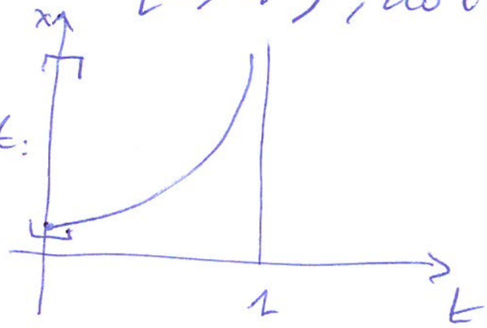
$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$




$\frac{\partial f}{\partial x} = 2x$

- f is locally Lipschitz (even differentiable)
- Lipschitz in any compact
- not globally Lipschitz ($\frac{\partial f}{\partial x} \to \infty$ as $x \to \infty$)
- solution can be extended only to $[0, 1)$, not beyond

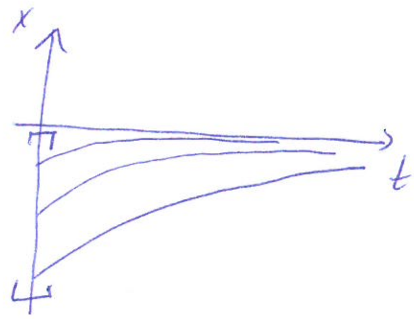
can take any compact: f always Lipschitz in any compact




as $t \to \infty$, $x(t)$ escapes any compact set

changing initial cond:

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$$




- if $x_0 > 0$ still finite escape time
- if $x_0 < 0$ solution can be extended $\forall t \to \infty$

reason: $x(t) \in W$ compact $W \subset \mathbb{R}_-$

W compact is also invariant

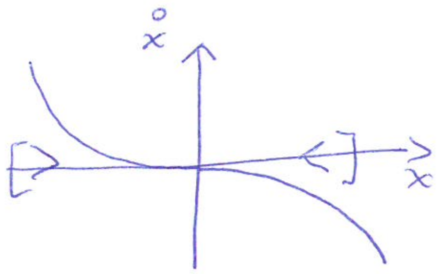
$\Rightarrow$ how much can the solution be extended depends on $x_0$! (and I need to know the solution $\Rightarrow$ unfeasible in general).

existence and uniqueness in compact sets

<u>Thm</u> If $f(t,x)$ cont in $x$ in $\mathcal{D}$, piecewise cont in $t$ in $[t_0, t_1]$, locally Lipschitz in $[t_0, t_1] \times \mathcal{D}$ and if $\exists\ W \subset \mathcal{D}$ compact and invariant for the system, then $x_0 \in W \Rightarrow$ the solution $\exists$ unique in $[t_0\ t_1]$

<u>def</u> $W$ is an <u>invariant set</u> for $\dot{x} = f(t,x)$ if $x_0 \in W$ $\Rightarrow x(t) \in W \quad \forall t$

<u>example</u> $\begin{cases} \dot{x} = -x^3 \\ x(0) = x_0 \end{cases}$




$f(x)$ is Locally Lipschitz on $\mathbb{R}$
solution never leaves the compact set $\{x \in \mathbb{R}\ \text{s.t.}\ |x| \leq |x_0|\}$
$\Rightarrow$ solution can be extended till $t \to \infty$

<u>Lemma</u> If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ cont. in $[t_0, t_1] \times \mathbb{R}^n$ then $f$ is globally Lipschitz $\Longleftrightarrow$ $\frac{\partial f}{\partial x}$ is uniformly bounded on $[t_0, t_1] \times \mathbb{R}^n$

# STABILITY ANALYSIS

Consider an autonomous system (without control)

$$\dot{x} = f(x) \qquad x \in \mathcal{D} \subset \mathbb{R}^n \qquad \mathcal{D} \text{ open}$$

equilibrium point can always be considered the origin $\bar{x} = 0$

In fact, if $\bar{x} \neq 0$, call $z = x - \bar{x}$

$$\dot{z} = \dot{x} = f(z - \bar{x}) = \bar{f}(z)$$

with $\bar{z} = 0$ as equil point

def $\bar{x} = 0$ equil point is

- stable if for each $\epsilon > 0$ $\exists\, \delta > 0$ s.t.

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$$

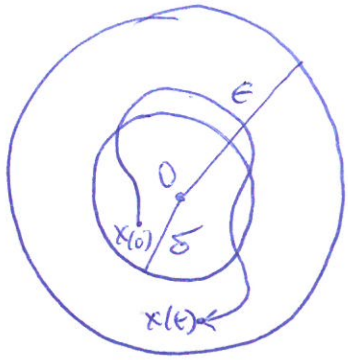
- unstable if it is not stable

- (locally) asymptotically stable if it is stable and $\delta$ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

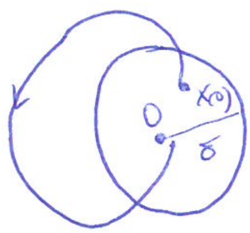
meaning

- stability




provided that you start in $B(0,\delta)$ you never exit the ball $B(0,\epsilon)$

- asymptotic stability : stability + convergence




convergence : provided you start inside the ball $B(0,\delta)$ you eventually converge to 0

# Lyaponov direct method

Main idea: ~~find a~~ given $\begin{cases} \dot{x} = f(x) \\ x(0) = 0 \end{cases}$ find a function

V for which $\bar{x} = 0$ is a local minimum
(V is called <u>Lyapunov function</u>)

For asymptotic stability we would like to have

- $V : \mathcal{D} \to \mathbb{R}$ which is $C^1(\mathcal{D})$
- $V(0) = 0$
- $V(x) > 0$ in $\mathcal{D} \setminus \{0\}$ → <u>positive definite function</u>
- derivatives of V along the trajectories of the system decrease (i.e. solutions of the system move towards the local min of V)

$$\dot{V}(x(t)) = \underbrace{\frac{\partial V}{\partial x}}_{\text{gradient}} \dot{x} = \frac{\partial V}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0$$

if $f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$

$\dot{V}(x) \leq 0$ <u>negative semidefinite function</u>
$(\dot{V}(0) = 0)$

classical examples:

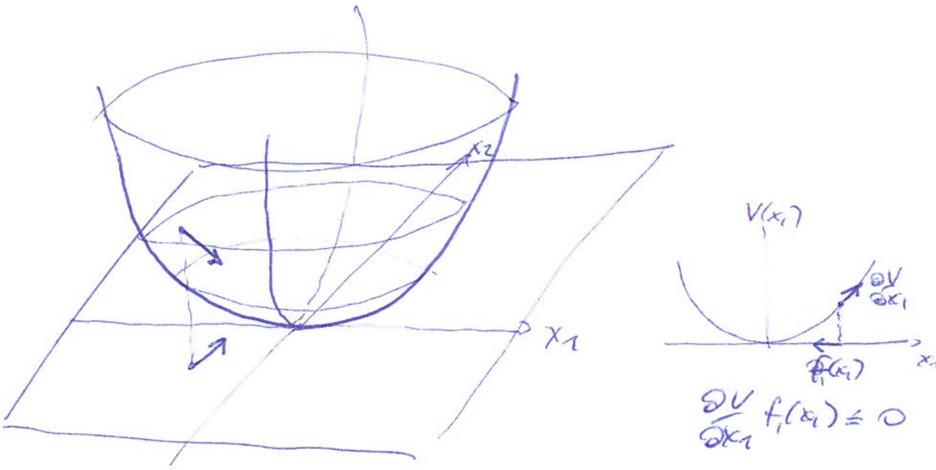
- energy function or potential energy of a mechanical system

- quadratic functions of linear systems

$$V(x) = x^T P x \qquad P = P^T > 0 \quad \text{posit. def.}$$

- picture in $\mathbb{R}^2$

$\mathring{V}(x)$ decreases iff $x$ "gets closer" to $\bar{x} = 0$




- Key point: avoid solving the ODE ($\mathring{V}(x)$ is computed along the trajectories but since

$$\mathring{V}(x) = \frac{\partial V}{\partial x} f(x),$$

I do not need the explicit solution to do so)

- $\mathring{V}(x) \leq 0 \Rightarrow x(t)$ stays in a compact + Lipschitz
  - $\Rightarrow$ solution can be prolonged $\forall\, t > 0$
  - $\Rightarrow$ solution that stays in an arbitrarily small compact around $\bar{x} = 0$ is stable (informally)