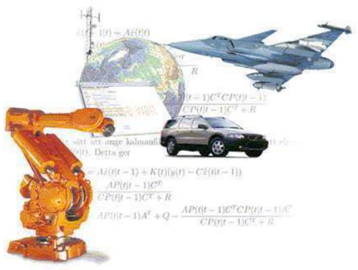
Robust Multivariable Control

Lecture 6



Anders Helmersson
anders.helmersson@liu.se

ISY/Reglerteknik
Linköpings universitet

Today's topics

- Parrott's Theorem
- LMIs
- Elimination Lemma
- H_∞ synthesis using LMIs
- Bounded Real Lemma
- Synthesis

Parrott's Theorem

Consider the optimization problem

$$\inf_X \left\| \begin{bmatrix} A & B \\ C & X \end{bmatrix} \right\|$$

This can be rewritten as

$$\inf_{\gamma, X} \gamma \quad \text{subject to} \quad \left[\begin{array}{cc|cc} -\gamma & 0 & A & B \\ 0 & -\gamma & C & X \\ \hline A^* & C^* & -\gamma & 0 \\ B^* & X^* & 0 & -\gamma \end{array}\right] \prec 0$$

Parrott's Theorem

Reorder rows and columns

$$\left[\begin{array}{cc|cc} -\gamma & A & & B \\ A^* & -\gamma & C^* & \\ \hline & C & -\gamma & X \\ B^* & & X^* & \gamma \end{array}\right] \prec 0$$

Apply Schur complement

$$\begin{bmatrix} -\gamma & X \\ X^* & \gamma \end{bmatrix} - \begin{bmatrix} & C \\ B^* & \end{bmatrix} \begin{bmatrix} -\gamma & A \\ A^* & -\gamma \end{bmatrix}^{-1} \begin{bmatrix} & B \\ C^* & \end{bmatrix} \prec 0$$

Parrott's Theorem

We can clear the off-diagonal elements by letting

$$X = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} -\gamma & A \\ A^* & -\gamma \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}$$

or, as the rank-deficient solution to

$$\arg\min_X \operatorname{rank} \begin{bmatrix} & C & X \\ -\gamma & A & B \\ A^T & -\gamma & \end{bmatrix}$$

Parrott's Theorem

The remaining diagonal elements become

$$-\gamma - \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} -\gamma & A \\ A^* & -\gamma \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ C^* \end{bmatrix} \prec 0$$

and

$$-\gamma - \begin{bmatrix} B & 0 \end{bmatrix} \begin{bmatrix} -\gamma & A \\ A^* & -\gamma \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \prec 0$$

which are equivalent to

$$\begin{bmatrix} -\gamma & & A \\ & -\gamma & C \\ A^* & C^* & -\gamma \end{bmatrix} \prec 0 \quad \text{and} \quad \begin{bmatrix} -\gamma & A & B \\ A^* & -\gamma & \\ B^* & & -\gamma \end{bmatrix} \prec 0$$

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Parrott's Theorem

which in turn are equivalent to

$$\max\left\{\left\|\begin{bmatrix} A \\ C \end{bmatrix}\right\|, \left\|\begin{bmatrix} A & B \end{bmatrix}\right\|\right\} < \gamma$$

Note that this approach also work on operators in a Hilbert space.

LMIs

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i$$

where $F_i = F_i^T$

Let $X = X^T \succ 0$ denote a symmetric, positive definite matrix.

The set of x such that $F(x) \succ 0$ is convex:
If $X_1 \succ 0$ and $X_2 \succ 0$ then also

$$X_1 + \lambda(X_2 - X_1) = (1-\lambda)X_1 + \lambda X_2 \succ 0$$

Maximum singular value

$$\bar{\sigma}(X) = \|X\| < \gamma$$

is equivalent to

$$\begin{bmatrix} -\gamma I & X \\ X^T & -\gamma I \end{bmatrix} \prec 0.$$

In the sequel, I drop the identity matrix and write this as

$$\begin{bmatrix} -\gamma & X \\ X^T & -\gamma \end{bmatrix} \prec 0.$$

Schur complement

$$\begin{bmatrix} S & G^T \\ G & R \end{bmatrix} \prec 0$$

is equivalent to

$$S - G^T R^{-1} G \prec 0$$
$$R \prec 0$$

since, by using a congruence transformation,

$$\begin{bmatrix} I & -G^T R^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} S & G^T \\ G & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1} G & I \end{bmatrix} = \begin{bmatrix} S - G^T R^{-1} G & 0 \\ 0 & R \end{bmatrix} \prec 0$$

Elimination Lemma

When can we find a solution to

$$Q + UKV^T + VK^TU^T \prec 0?$$

What conditions on $Q = Q^T$, U and V are required in order to find a solution, K, that satisfies the matrix inequality?

We start by looking at a special case:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} + K \\ Q_{12}^T & Q_{22} & Q_{23} \\ (Q_{13} + K)^T & Q_{23}^T & Q_{33} \end{bmatrix} \prec 0$$

Note that this requires $Q_{22} \prec 0$.

Elimination Lemma

Use the Schur complement:

$$\begin{bmatrix} Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T & Q_{13} - Q_{12}Q_{22}^{-1}Q_{23} + K \\ Q_{13}^T - Q_{23}^T Q_{22}^{-1} Q_{12}^T + K^T & Q_{33} - Q_{23}^T Q_{22}^{-1} Q_{23} \end{bmatrix} \prec 0$$

where $Q_{22} \prec 0$.

We can zero the off-diagonal blocks by choosing $K = Q_{12}Q_{22}^{-1}Q_{23} - Q_{13}$:

$$\begin{bmatrix} Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T & 0 \\ 0 & Q_{33} - Q_{23}^T Q_{22}^{-1} Q_{23} \end{bmatrix} \prec 0$$

The diagonal blocks lead to two conditions, which in turn can be rewritten using Schur complement.

Elimination Lemma

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \prec 0$$

and

$$\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23}^T & Q_{33} \end{bmatrix} \prec 0$$

which are exactly those blocks that remain if we delete the rows and columns containing K (and K^T).

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13}+K \\ Q_{12}^T & Q_{22} & Q_{23} \\ (Q_{13}+K)^T & Q_{23}^T & Q_{33} \end{bmatrix} \prec 0$$

The General Elimination Lemma

The problem

$$Q + UKV^T + VK^T U^T \prec 0.$$

has a solution K if and only if

$$U_\perp Q U_\perp^T \prec 0,$$

and

$$V_\perp Q V_\perp^T \prec 0$$

hold.

Bounded Real Lemma

Introduce $V(x) = x^T P x$:

$$
\begin{aligned}
&\dot{V} + \gamma^{-1} y^T y - \gamma u^T u \\
&\quad = x^T (A^T P + PA + \dot{P}) x + x^T P B u + u^T B^T P x \\
&\qquad + \gamma^{-1} (Cx + Du)^T (Cx + Du) - \gamma u^T u \\
&\quad = \begin{bmatrix} x \\ u \end{bmatrix}^T \left(\begin{bmatrix} A^T P + PA + \dot{P} & PB \\ B^T P & -\gamma \end{bmatrix} + \gamma^{-1} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix}
\end{aligned}
$$

Bounded Real Lemma

Integrate:

$$\int_0^\infty \dot{V} + \gamma^{-1} y^T y - \gamma u^T u \, dt =$$

$$= V(\infty) - V(0)$$

$$+ \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \underbrace{\left(\begin{bmatrix} A^T P + PA + \dot{P} & PB \\ B^T P & -\gamma \end{bmatrix} + \gamma^{-1} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right)}_{\prec 0} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

Apply Schur complement:

$$\begin{bmatrix} A^T P + PA + \dot{P} & PB & C^T \\ B^T P & -\gamma & D^T \\ C & D & -\gamma \end{bmatrix} \prec 0$$

Anders Helmersson anders.helmersson@liu.se

Bounded Real Lemma

$\|G\|_\infty < \gamma$ is equivalent to that there exists $P = P^T \succ 0$, such that

$$\begin{bmatrix} PA + A^T P + \dot{P} & PB & C^T \\ B^T P & -\gamma & D^T \\ C & D & -\gamma \end{bmatrix} \prec 0$$

H_∞ synthesis

Consider the system

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

with the controller

$$K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

in a feedback loop. If we assume that $D_{22} = 0$, we obtain

$$\tilde{G} = G \star K = \left[\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_{11} \end{array}\right] + \left[\begin{array}{cc} B_2 & 0 \\ 0 & I \\ \hline D_{12} & 0 \end{array}\right] \left[\begin{array}{cc} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{array}\right] \left[\begin{array}{cc|c} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{array}\right]$$

H_∞ synthesis

Apply the Bounded Real Lemma:

$$\begin{bmatrix} XA+A^TX+\dot{X} & A^TN+\dot{N} & XB_1 & C_1^T \\ N^TA+\dot{N}^T & \dot{L} & N^TB_1 & 0 \\ B_1^TX & B_1^TN & -\gamma & D_{11}^T \\ C_1 & 0 & D_{11} & -\gamma \end{bmatrix}$$

$$+\begin{bmatrix} XB_2 & N \\ N^TB_2 & L \\ 0 & 0 \\ D_{12} & 0 \end{bmatrix} \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} \begin{bmatrix} C_2 & 0 & D_{21} & 0 \\ 0 & I & 0 & 0 \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}^T \prec 0$$

where

$$P = \begin{bmatrix} X & N \\ N^T & L \end{bmatrix} = \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix}^{-1}.$$

H_∞ synthesis

Move the second row and column to the last position:

$$\begin{bmatrix} XA + A^T X + \dot{X} & XB_1 & C_1^T & A^T N + \dot{N} \\ B_1^T X & -\gamma & D_{11}^T & B_1^T N \\ C_1 & D_{11} & -\gamma & 0 \\ N^T A + \dot{N}^T & N^T B_1 & 0 & \dot{L} \end{bmatrix}$$

$$+ \begin{bmatrix} XB_2 & N \\ 0 & 0 \\ D_{12} & 0 \\ N^T B_2 & L \end{bmatrix} \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}^T \prec 0$$

where

$$P = \begin{bmatrix} X & N \\ N^T & L \end{bmatrix} = \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix}^{-1}.$$

H_∞ synthesis

Apply the elimination lemma:

$$\begin{bmatrix} & \end{bmatrix}^T \begin{bmatrix} XA + A^TX + \dot{X} & XB_1 & C_1^T & A^TN + \dot{N} \\ B_1^TX & -\gamma & D_{11}^T & B_1^TN \\ C_1 & D_{11} & -\gamma & 0 \\ N^TA + \dot{N}^T & N^TB_1 & 0 & \dot{L} \end{bmatrix} \begin{bmatrix} \mathcal{N}_{X1} & 0 \\ \mathcal{N}_{X2} & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & \end{bmatrix}^T \begin{bmatrix} XA + A^TX + \dot{X} & XB_1 & C_1^T \\ B_1^TX & -\gamma & D_{11}^T \\ C_1 & D_{11} & -\gamma \end{bmatrix} \begin{bmatrix} \mathcal{N}_{X1} & 0 \\ \mathcal{N}_{X2} & 0 \\ 0 & I \end{bmatrix} \prec 0$$

where

$$\begin{bmatrix} C_2 & D_{21} & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{N}_{X1} & 0 \\ \mathcal{N}_{X2} & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} = 0$$

H_∞ synthesis

In the same way, we can rewrite the second condition as

$$\begin{bmatrix} \quad \end{bmatrix}^T \begin{bmatrix} XA + A^TX + \dot{X} & A^TN + \dot{N} & XB_1 & C_1^T \\ N^TA + \dot{N}^T & \dot{L} & N^TB_1 & 0 \\ B_1^TX & B_1^TN & -\gamma & D_{11}^T \\ C_1 & 0 & D_{11} & -\gamma \end{bmatrix} \begin{bmatrix} Y & M & 0 & 0 \\ M^T & * & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{N}_{Y1} & 0 \\ 0 & I \\ 0 & 0 \\ \mathscr{N}_{Y2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \quad \end{bmatrix}^T \begin{bmatrix} AY + YA^T - \dot{Y} & YC_1^T & B_1 \\ C_1Y & -\gamma & D_{11} \\ B_1^T & D_{11}^T & -\gamma \end{bmatrix} \begin{bmatrix} \mathscr{N}_{Y1} & 0 \\ \mathscr{N}_{Y2} & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \prec 0$$

where

$$\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix} \begin{bmatrix} \mathscr{N}_{Y1} \\ \mathscr{N}_{Y2} \end{bmatrix} = 0$$

H_∞ synthesis

Thus

$$
\begin{bmatrix} \mathscr{N}_X & 0 \\ 0 & I \end{bmatrix}^T
\left[\begin{array}{cc|c} XA + A^T X + \dot{X} & XB_1 & C_1^T \\ B_1^T X & -\gamma & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma \end{array}\right]
\begin{bmatrix} \mathscr{N}_X & 0 \\ 0 & I \end{bmatrix} \prec 0
$$

$$
\begin{bmatrix} \mathscr{N}_Y & 0 \\ 0 & I \end{bmatrix}^T
\left[\begin{array}{cc|c} AY + YA^T - \dot{Y} & YC_1^T & B_1 \\ C_1 Y & -\gamma & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma \end{array}\right]
\begin{bmatrix} \mathscr{N}_Y & 0 \\ 0 & I \end{bmatrix} \prec 0
$$

where X and Y are connected by

$$
P = \begin{bmatrix} X & N \\ N^T & L \end{bmatrix} = \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix}^{-1}.
$$

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$$P = \begin{bmatrix} X & N \\ N^T & L \end{bmatrix} = \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix}^{-1},$$

from which we get $Y^{-1} = X - NL^{-1}N^T$.

Thus $X - Y^{-1} = NL^{-1}N^T \succeq 0$. Next, use the Schur complement:

$$X - Y^{-1} \succeq 0 \quad \Leftrightarrow \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$$

In addition, if $X - Y^{-1} \succeq 0$ then we can reverse the argument and find a P. Perform Cholesky factorization on $X - Y^{-1} = NN^T$ and choose $L = I$, which gives $P \in \mathbb{R}^{(n+r)\times(n+r)}$ where $r = \operatorname{rank}(XY - I)$.

The controller has r states.

H_∞ synthesis – summary

Conditions for the existence of a controller:

$$\begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} XA + A^T X + \dot{X} & XB_1 & C_1^T \\ B_1^T X & -\gamma & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma \end{array}\right] \begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AY + YA^T - \dot{Y} & YC_1^T & B_1 \\ C_1 Y & -\gamma & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma \end{array}\right] \begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix} \prec 0$$

where $\mathcal{N}_X$ and $\mathcal{N}_Y$ span the null space of $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ and $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$, respectively.

X and Y are connected by $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$

and $r = \operatorname{rank}(XY - I)$ is the order of the controller.

H_∞ synthesis – properties

- We obtain three LMIs that specify conditions on X_{LMI} and Y_{LMI} as functions of γ.
- Compare with the traditional solution: two Riccati equations and one condition $\rho(X_\infty Y_\infty) < \gamma^2$.
- There are no other conditions on the system's matrices.
- The order of the controller, r, specifies a nonlinear condition (not a convex problem): difficult to solve!

Connection: $X_{\text{LMI}}^{-1} \sim \gamma Y_\infty$ (observer) and $Y_{\text{LMI}}^{-1} \sim \gamma X_\infty$ (controller).