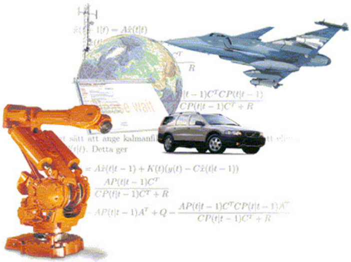
Robust Multivariable Control

Lecture 4



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Today's topics

- LQR
- The Riccati equation
- Coprime factorization
- Youla parametrization

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LQR – Linear Quadratic Regulator

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

We want to minimize

$$\int_0^\infty y^T(t)y(t) + u^T(t)u(t)dt = \|y\|_2^2 + \|u\|_2^2 \tag{1}$$

Introduce the Riccati equation

$$A^T S + SA - SBB^T S + C^T C = 0 \tag{2}$$

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LQR – Linear Quadratic Regulator

Introduce $V(x) = x^T S x$:

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T S x + x^T S \dot{x} \\
&= x^T A^T S x + u^T B^T S x + x^T S A x + x^T S B u \\
&= u^T B^T S x + x^T S B u + x^T \underbrace{\left(A^T S + SA\right)}_{(2):\ SBB^T S - C^T C} x \\
&= u^T B^T S x + x^T S B u + x^T S B B^T S x - \underbrace{x^T C^T C x}_{y^T y} \\
&= \left(u + B^T S x\right)^T \left(u + B^T S x\right) - u^T u - y^T y
\end{aligned}$$

LQR

Integrate $\dot{V}(x)$:

$$\begin{aligned}\int_0^\infty \dot{V}(x)dt &= V(x(\infty)) - V(x(0)) \\ &= -\|y\|_2^2 - \|u\|_2^2 + \|u + B^T Sx\|_2^2\end{aligned}$$

Thus,

$$\|y\|_2^2 + \|u\|_2^2 = \|u + B^T Sx\|_2^2 + V(x(0)) - \underbrace{V(x(\infty))}_{\to 0 \text{ if stable}}$$

The minimum is obtained for $u = -B^T Sx$.

The closed loop dynamics is described by $A - BB^T S$.

We want to obtain stable eigenvalues in the LHP.

The Riccati equation

$$A^T S + SA - SBB^T S + C^T C = 0$$

or, more generally,

$$A^T X + XA + XRX + Q = 0 \tag{3}$$

has $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ solutions.

When $n = 1$ we obtain 2 solutions.
When $n = 2$ we obtain 6 solutions.
When $n = 5$ we obtain 252 solutions.
When $n = 10$ we obtain 184756 solutions!

Which solution to choose?

We want $A + RX$ (or $A - BB^T S$) to become stable.

An example

$$A = \begin{bmatrix} -2 & 0.3 \\ 0.2 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.7 \\ -0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.7 & -0.1 \end{bmatrix}$$

In Matlab: `X = are (A, B*B', C'*C);`
Six solutions:

$$\begin{bmatrix} 0.1195 & -0.0114 \\ -0.0114 & 0.0017 \end{bmatrix}, \begin{bmatrix} 9.579 & 52.833 \\ -2.396 & -13.319 \end{bmatrix}, \begin{bmatrix} -9.611 & -54.37 \\ -54.37 & -303.63 \end{bmatrix},$$

$$\begin{bmatrix} -7.392 & 1.882 \\ 1.882 & -0.4756 \end{bmatrix}, \begin{bmatrix} 9.579 & -2.396 \\ 52.833 & -13.319 \end{bmatrix}, \begin{bmatrix} -178.3 & -511.1 \\ -511.1 & -1540.6 \end{bmatrix}$$

The solutions are not necessarily real, symmetric or positive definite.

The Hamiltonian

Introduce the Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{4}$$

Assume that λ is an eigenvalues of H.
If H is real then also $\bar{\lambda}$ is an eigenvalue.

Also $-\bar{\lambda}$ is an eigenvalue of H since

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad J^2 = -I, \qquad J^{-1}HJ = -JHJ = -H^T$$

Thus, H and $-H^T$ are similar (have the same eigenvalues).

The eigenvalues are symmetric with respect to the imaginary and real axes.

Eigenvectors

Choose n (out of $2n$ possible) eigenvectors and stack them together:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Assume that there are no eigenvalues on the imaginary axis:

$$H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Lambda$$

Assume further that $x_1 \in \mathbb{R}^{n \times n}$ is invertible:

$$H \begin{bmatrix} I \\ x_2 x_1^{-1} \end{bmatrix} = H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} x_1^{-1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} x_1^{-1} \underbrace{x_1 \Lambda x_1^{-1}}_{\tilde{\Lambda}} = \begin{bmatrix} I \\ x_2 x_1^{-1} \end{bmatrix} \tilde{\Lambda}$$

Eigenvalues

Let $X = x_2 x_1^{-1}$:

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \tilde{\Lambda} \tag{5}$$

We then get

$$\begin{bmatrix} -X & I \end{bmatrix} \underbrace{\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} I \\ X \end{bmatrix} = -XA - A^T X - XRX - Q = 0$$

and

$$A + RX = \begin{bmatrix} A & R \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \tilde{\Lambda}$$

according to the first row in (5).
Thus, $A + RX$ has the same eigenvalues as Λ (and $\tilde{\Lambda}$).

Stable solutions

Choose eigenvectors

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

such that Λ becomes stable (eigenvalues in LHP).

Notation: $H \in \mathrm{dom}(Ric)$ if

- H has no imaginary eigenvalues;
- x_1 nonsingular.

$X^T = X = \mathrm{Ric}(H) = x_2 x_1^{-1}$.

Symmetry, $X = X^T$

Subtract (3) from its transpose:

$$
\begin{aligned}
&(A^T X + XA + XRX + Q)^T - (A^T X + XA + XRX + Q) \\
&= (X^T - X)A + A^T(X^T - X) + X^T R X^T - XRX \\
&= (X^T - X)(A + RX) + (A + RX)^T (X^T - X) = 0
\end{aligned}
$$

This is a Lyapunov equation in $X^T - X$.
It has a unique zero solution if $A + RX$ is stable.
More generally, if $A + RX$ and $-(A + RX)$ has no common eigenvalues.

Extremal solutions

Let

$$\mathscr{Q}(X) = A^T X + XA + XRX + Q$$

where $R \preceq 0$ and $X = X^T$.

If (A, R) is stabilizable, then there exists a unique maximal solution X_+ to the Riccati equation, $\mathscr{Q}(X) = 0$.

Furthermore $X_+ \succeq X$, $\forall X$ such that $\mathscr{Q}(X) \succeq 0$.

Also, $A + RX_+$ has its eigenvalues in $\bar{\mathbb{C}}_-$.

There also exists a similar minimal solution, X_-, to $\mathscr{Q}(X) \succeq 0$.

Generalized Eigenvalue Problem

The eigenvalues are given by

$$\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\underbrace{\begin{bmatrix} R & A \\ A^T & Q \end{bmatrix}}_{\text{symmetric}} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} & I \\ -I & \end{bmatrix}}_{\text{anti-symmetric}} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

or

$$\begin{bmatrix} R & A - \lambda I \\ A^T + \lambda I & Q \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

Generalized Eigenvalue Problem

Specifically for the LQR problem:

$$\begin{bmatrix} -BB^T & A-\lambda I \\ A^T+\lambda I & C^T C \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} -BB^T & A-\lambda I & \\ A^T+\lambda I & & C^T \\ & C & -I \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ y \end{bmatrix} = 0$$

or

$$\left[\begin{array}{cc|cc} & & A-\lambda I & B \\ & -I & C & \\ \hline A^T+\lambda I & C^T & & \\ B^T & & & I \end{array}\right] \left[\begin{array}{c} x_2 \\ y \\ \hline x_1 \\ u \end{array}\right] = 0$$

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Coprime factorization

Used for “loop-shaping”. Circumvents problems with unstable system or zeros on the imaginary axis.

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

There are two types of factorizations:

$$G = NM^{-1} \qquad \text{RCF (right coprime factorization)} \tag{7}$$

and

$$G = \tilde{M}^{-1}\tilde{N} \qquad \text{LCF (left coprime factorization)} \tag{8}$$

We want

$$\begin{bmatrix} M \\ N \end{bmatrix} \in RH_\infty$$

Coprime

M and N are coprime if there exist X and $Y \in RH_\infty$ such that the Bezout's equation holds

$$XM + YN = I \tag{9}$$

For instance in the ring of polynomials, let $M = s+1$ and $N = s^2+1$. Choose $X = \frac{1-s}{2}$ and $Y = \frac{1}{2}$, which implies

$$\frac{1-s}{2}(s+1) + \frac{1}{2}(s^2+1) = 1$$

Is this always possible?

No! Choose for instance $M = s+1$ and $N = s^2 - 1 = (s+1)(s-1)$, which have a common factor.

Stabilizing

Let G be defined by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Find an F such that $A + BF$ becomes stable.

This is possible if and only if (A, B) is stabilizable.

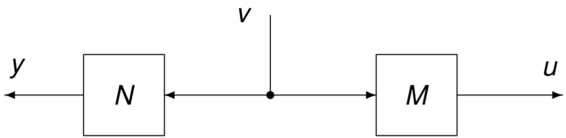
Introduce $v = u - Fx$ such that $u = Fx + v$.

$$\begin{aligned} \dot{x} &= (A + BF)x + Bv \\ u &= Fx + v \\ y &= (C + DF)x + Dv \end{aligned}$$

Find a factorization

We claim that

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right] \in RH_\infty$$

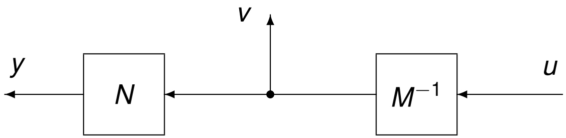


Inverting

Note that M is invertible since

$$M^{-1} = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} A+BF-BF & -B \\ \hline F & I \end{array}\right] = \left[\begin{array}{c|c} A & -B \\ \hline F & I \end{array}\right]$$

The D matrix is invertible!



Coprime

Choose L such that $A+LC$ becomes stable (Hurwitz).

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \end{array}\right] \in RH_\infty$$

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} X_r & Y_r \end{bmatrix} \times \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right] \in RH_\infty$$

$$= \left[\begin{array}{cc|c} A+BF & 0 & B \\ -(B+LD)F+L(C+DF) & A+LC & -(B+LD)+LD \\ \hline F & F & I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A+BF & 0 & B \\ LC-BF & A+LC & -B \\ \hline F & F & I \end{array}\right]$$

Similarity transformations

Choose an appropriate similarity transformation matrix according to

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}^{-1}$$

$$\left[\begin{array}{cc|c} A+BF & 0 & B \\ LC-BF & A+LC & -B \\ \hline F & F & I \end{array}\right] \sim \left[\begin{array}{cc|c} A+BF & 0 & B \\ 0 & A+LC & 0 \\ \hline 0 & F & I \end{array}\right] \sim I$$

Thus, Bezout's equation is satisfied, which means that M and N are coprime.

How can we choose F?

Normalized coprime factorization (NCF)

Choose M and N such that

$$M^{\sim}M + N^{\sim}N = M^T(-s)M(s) + N^T(-s)N(s) = I,$$

which is equivalent to

$$\begin{bmatrix} M \\ N \end{bmatrix}^{\sim} \begin{bmatrix} M \\ N \end{bmatrix} = I \quad \Leftrightarrow \quad \left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\|_{\infty} = 1$$

Here we choose F according to a Riccati equation, see §13.8 in ZDG.

In Matlab we can use `rncf` and `lncf`.

Normalized right coprime factorization

Let $R = I + D^T D$, $\tilde{R} = I + DDi^T$ and

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A + BF & BR^{1/2} \\ \hline F & R^{1/2} \\ C + DF & DR^{1/2} \end{array}\right]$$

where $F = -R^{-1}(B^T X + D^T C)$ and

$$X = \mathrm{Ric}\begin{bmatrix} A - BR^{-1}D^T C & -BR^{-1}B^T \\ -C^T \tilde{R}^{-1} C & -\left(A - BR^{-1}D^T C\right)^T \end{bmatrix}$$

Example of an NCF

$$G(s) = \frac{k}{s} = \underbrace{\frac{k}{s+k}}_{N} \underbrace{\left(\frac{s}{s+k}\right)^{-1}}_{M^{-1}} = NM^{-1}$$

Coprime?

$$1 \times \frac{s}{s+k} + 1 \times \frac{k}{s+k} = 1$$

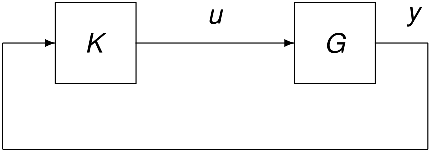
Normalized?

$$M^T(-s)M(s) + N^T(-s)N(s)$$
$$= \frac{-s^2}{(-s+k)(s+k)} + \frac{k^2}{(-s+k)(s+k)} = 1$$

Youla parametrization

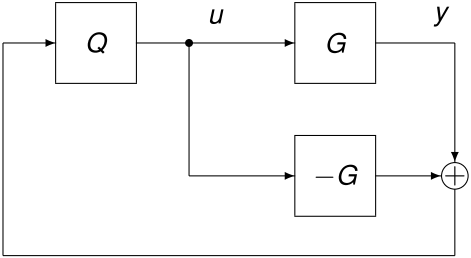
A Youla parametrization parametrizes all stabilizing controllers to a system G.

Suppose that the system G is stable.



Youla parametrization

We want to parametrize all stabilizing controllers K:

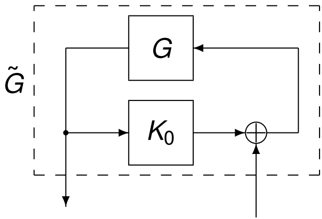


$$K = Q(I + GQ)^{-1}, \qquad Q = K(I - GK)^{-1}, \qquad Q \in RH_\infty$$

This is also called IMC (Internal Model Control).

G unstable

Assume that the system G is unstable.
First find a K_0 that stabilizes G.



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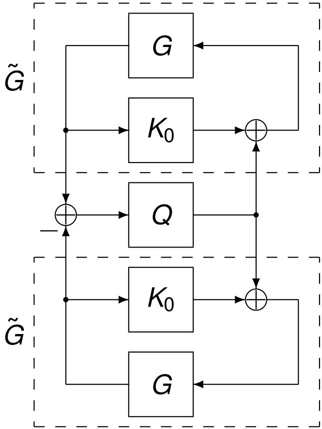

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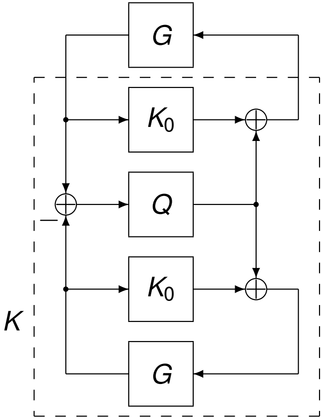

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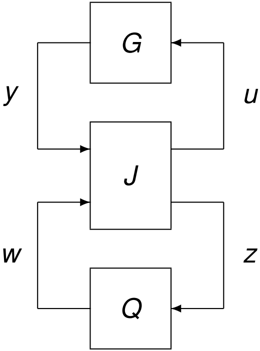
G unstable

A stable Q always generates a stabilizing controller K. But,

$$\tilde{G} = G(I - K_0 G)^{-1}$$
$$K = K_0 + Q(I + \tilde{G}Q)^{-1}$$
$$Q = \left(I - (K - K_0)\tilde{G}\right)^{-1}(K - K_0)$$
$$Q = (I - K_0 G)(I - KG)^{-1}(K - K_0)$$

If K_0 is unstable then the unstable poles may appear in Q even if G and K in a closed loop are stable. Thus, all stabilizing controllers cannot be generated with this parametrization.

General



General

$$J = \left[\begin{array}{c|cc} A+BF+LC+LDF & -L & B+LD \\ \hline F & 0 & I \\ -(C+DF) & I & -D \end{array}\right]$$

General

G:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

J:

$$\dot{\bar{x}} = (A + BF + LC + LDF)\bar{x} - Ly + (B + LD)w$$
$$u = F\bar{x} + w$$
$$z = -(C + DF)\bar{x} + y - Dw$$

$G \star J$:

$$\dot{x} = Ax + BF\bar{x} + Bw$$
$$\dot{\bar{x}} = (A + BF + LC + LDF)\bar{x} - L(Cx + D(F\bar{x} + w)) + (B + LD)w$$
$$= -LCx + (A + BF + LC)\bar{x} + Bw$$
$$z = -C\bar{x} + Cx$$

General

Use $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$ for a similarity transformation

$$\left[\begin{array}{cc|c} A & BF & B \\ -LC & A+BF+LC & B \\ \hline C & -C & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} A+LC & 0 & 0 \\ -LC & A+BF & B \\ \hline C & 0 & 0 \end{array}\right]$$

This is a “zero” system (if we neglect transients). Thus, we can use this parametrization with an arbitrary, stable Q and maintain stability in the closed-loop system.