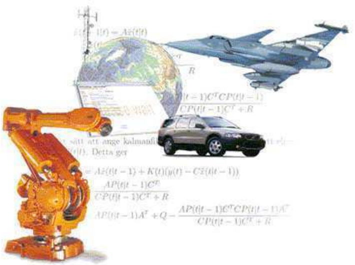
Robust Multivariable Control

Lecture 1



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https://users.isy.liu.se/rt/andersh/teaching/robkurs.html

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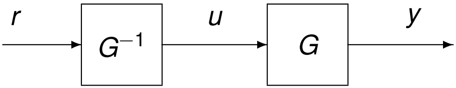
Feedback

Why do we need feedback?



We want the output y to follow the reference input r.

When *cannot* we do like this?

- G is unstable;
- G is uncertain;
- G is not minimum phase (zeros in RHP, G^{-1} unstable).

A failure – Conestoga 1620 rocket failure

https://www.youtube.com/watch?v=wWDJBkf_P3Yi

A success – SpaceX core stage landing

https://youtu.be/sYmQQn_ZSys

Simple dynamics

$$\frac{d}{dt}\begin{bmatrix}\dot{\theta}\\ \theta\\ \dot{y}\\ y\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & T/m & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \theta\\ \dot{y}\\ y\end{bmatrix} + \begin{bmatrix}-\ell T/I\\ 0\\ T/m\\ 0\end{bmatrix}\delta$$

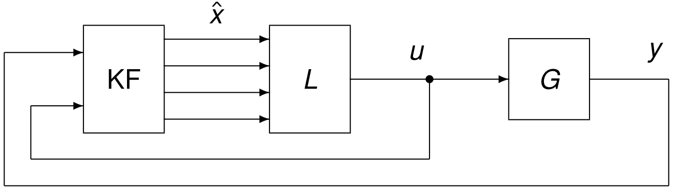
Attitude angle, θ, and lateral position, y, as outputs.
Nozzle deflection, δ, as input.

Essentially the same dynamics for a Lunar landing: two double integrators.

An example – SpaceX core stage landing

1. The same dynamics applies to both liftoff and landing.
2. What are the differences ?
3. Two, three or four integrators in series ?
4. How does this affect the robustness margin?
5. How can we handle several inputs and outputs?

Some history



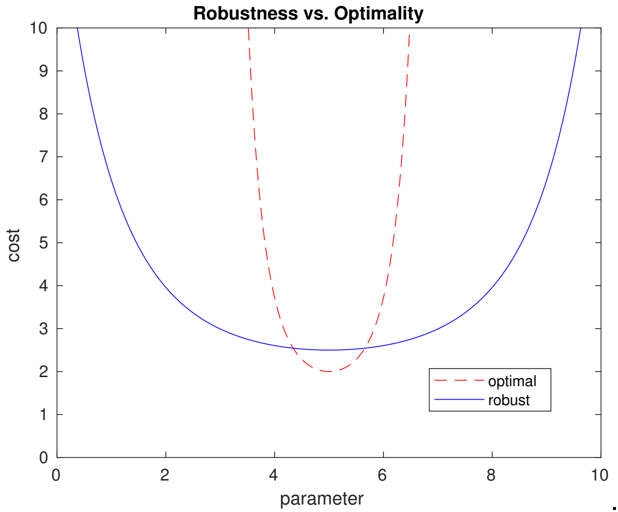
LQR/LQE is optimal in some sense

What about phase and amplitude margins?

Abstract: “There are none”, Doyle, 1978, chapter 14.10 in ZDG.

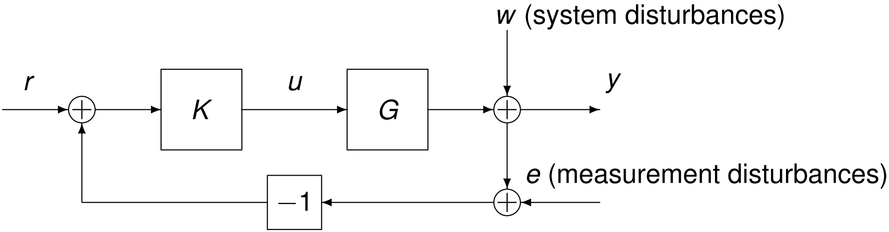
LTR gives margins (Loop Transfer Recovery).

Robustness



Robustness vs. Optimality
cost
parameter
0
2
4
6
8
10
0
1
2
3
4
5
6
7
8
9
10
optimal
robust

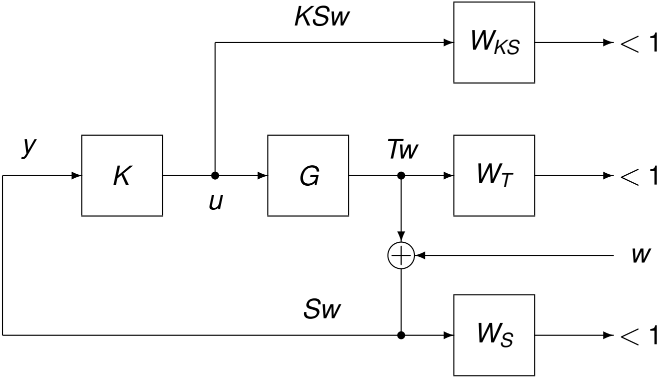
Standard form



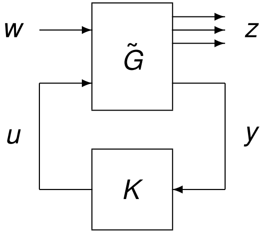
$$y = Sw + T(r - e)$$
$$S = (I + GK)^{-1}$$
$$T = GK(I + GK)^{-1}$$
$$S + T = (I + GK)(I + GK)^{-1} = I$$

Augmented system

We would also like to reduce the control signals in magnitude.



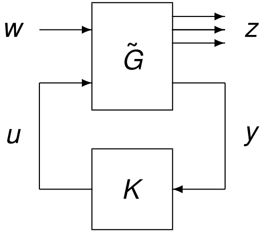
Augmented system



We want to limit the gain from w to z by a suitable choice of K.

Try to reduce the gain to one or below.

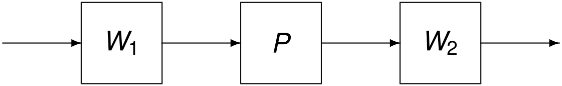
Design methods



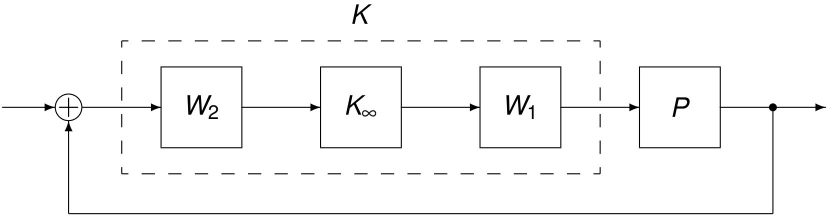
1) Specify the requirements
2) Synthesis
3) Check if the requirements are satisfied
4) Modify and repeat from 1)

Loop shaping

Find W_1 and W_2 to shape the open loop gain



The controller, K, can be obtained in a synthesis step



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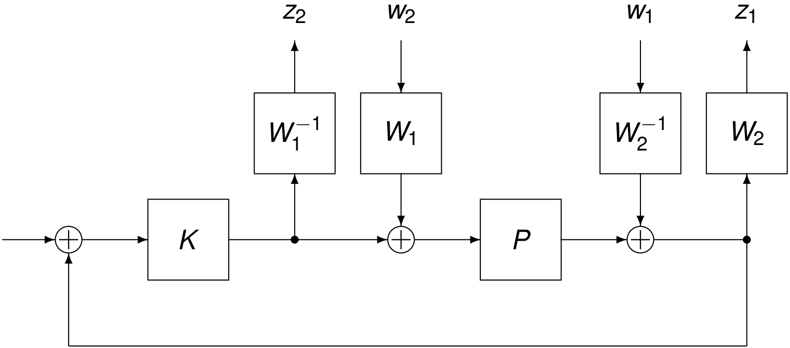
Alternative formulation



Multivariable gain

Consider

$$z = Mw$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0.96 & 1.72 \\ 2.28 & 0.96 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Let $\|w\|^2 = w_1^2 + w_2^2 = 1$, and maximize $\|z\|^2 = z_1^2 + z_2^2$.

Multivariable gain

Singular value decomposition:

$$M = \begin{bmatrix} 0.96 & 1.72 \\ 2.28 & 0.96 \end{bmatrix} = U\Sigma V^T$$

$$= \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}^T$$

where $U^T U = I$, $V^T V = I$, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \succeq 0$.

Use `svd` in Matlab.

Maximum gain, $\|M\| = \bar{\sigma}(M) = \bar{\sigma}(M^T) = \sigma_1 = \max_i \sigma_i$.

Different directions in w give different gains.

Connection to eigenvalues

$$M = U\Sigma V^T$$
$$M^T M = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$$

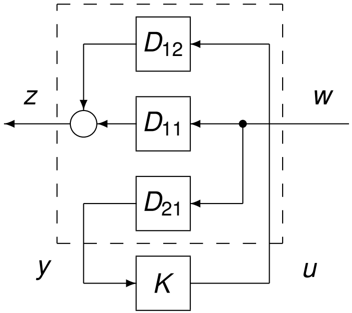
Thus, $M^T M V = V\Sigma^2$ and Σ^2 are eigenvalues to $M^T M$.

Also, the eigenvalues of $\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}$ are equal to $\pm\sigma_i$.

Computing the eigenvalues can be numerically difficult

Synthesis

Consider a static system:



Choose a K such that $\|D_{11} + D_{12}KD_{21}\|$ is minimized.

Synthesis

Use orthonormal transformations to obtain a similar problem

Find $\gamma = \min_K \left\| \begin{bmatrix} C & K \\ D & B \end{bmatrix} \right\|$.

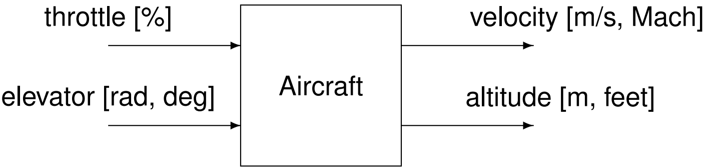
This is solved by Parrott's theorem:

$$\gamma = \max \left\{ \left\| \begin{bmatrix} D & B \end{bmatrix} \right\|, \left\| \begin{bmatrix} C \\ D \end{bmatrix} \right\| \right\}$$

One solution is $K = \arg\min_K \operatorname{rank} \begin{bmatrix} & C & K \\ -\gamma & D & B \\ D^T & -\gamma & \end{bmatrix}$

This problem is related to the elimination lemma.

Multivariable phenomena

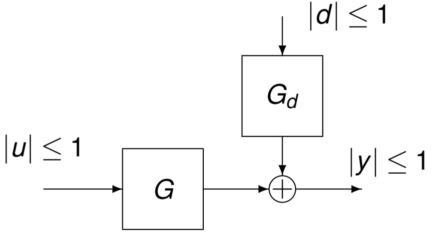


How can we compare rad and %?

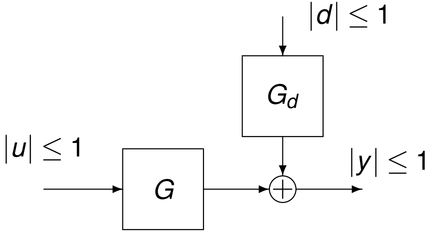
How can we compare m, m/s and Mach number?

Scaling

- Scale the input signals, u, so that they are limited between ± 1.
- Scale the output signals, y, so that they are limited ± 1.
- Scale the disturbances, d, so that they are limited ± 1.



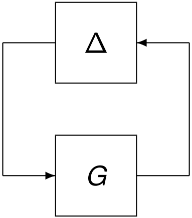
Scaling



In order to compensate for a disturbance so that $|y| \leq 1$:

$$\|G^{-1}G_d\| \leq 1$$

Small gain theorem

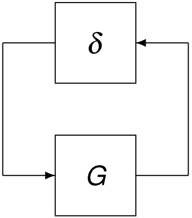


G stable and $\|G\| < 1$.

If $\|\Delta\| \leq 1$ then the closed loop system is stable

Small gain theorem

Pull out the uncertain parameters.
LFT = linear fractional transformation:



$$|\delta| \leq 1$$

Lectures

1. Introduction
2. Norms, Lyapunov equations, balancing
3. Feedback, stability, performance
4. Parametrization of regulators, Riccati equations, LQR
5. H_∞-synthesis
6. LMIs
7. Loop shaping, model reduction
8. Model uncertainties, small gain theorem, LFTs
9. μ analysis and synthesis
10. Summary