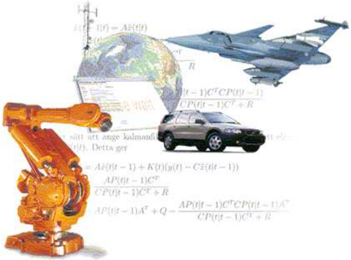
Robust Multivariable Control

Lecture 10



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Today's topic

- Repetition

Systems: Linear time-invariant (LTI)

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = D + C(sI - A)^{-1}B$$

Similarity transformations introduces new states, $x = T\hat{x}$:

$$G(s) = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$$

Inverse of systems

The inverse of a system can be written as

$$G^{-1}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

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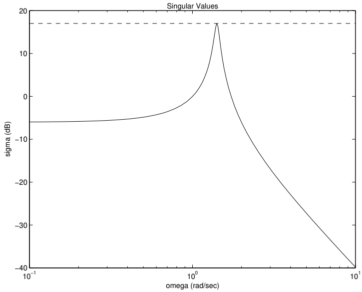
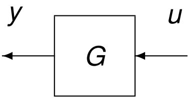
H_∞-norm

The H_∞-norm is the maximum (sup) of $\bar{\sigma}(G(j\omega))$ for stable systems.



Energy:



$$\|y\|_2 \leq \gamma \|u\|_2$$

$$\gamma = \|G\|_\infty$$

How to find $\|G\|_\infty$

Assume that $D = 0$:

$$H = \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

If $\gamma > \|G\|_\infty$, then H has no imaginary eigenvalues.

General case

$$H_\infty = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & D^T \\ D & -I \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^T \\ C & 0 \end{bmatrix}$$

Detectability and stabilizability

- A system is *detectable* if all unstable modes are observable
- A system is *stabilizable* if all unstable modes are controllable.

Balanced realization

A system is on balanced form if the controllability and observability Gramians are equal:

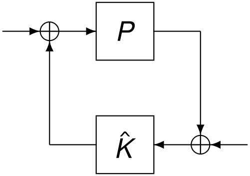
$$A^T\Sigma + \Sigma A + C^T C = 0$$

and

$$A\Sigma + \Sigma A^T + BB^T = 0.$$

Used for model reduction.

Stability

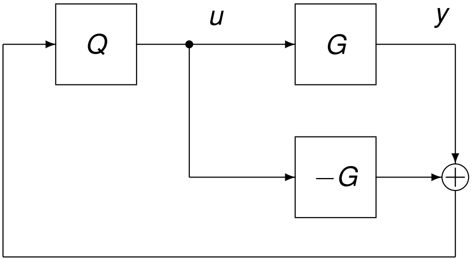


The closed loop system is stable if

$$\begin{bmatrix} I & \hat{K} \\ P & I \end{bmatrix}^{-1} \in RH_\infty$$

Youla parametrization

If G is stable then $Q \in RH_\infty$ parametrizes all stabilizing controllers:



$$K = Q(I + GQ)^{-1}$$
$$Q = K(I - GK)^{-1}$$
$$Q \in RH_\infty$$

Related to IMC = Internal model control.

Riccati equations

The Riccati equation

$$A^T X + XA + XRX + Q = 0$$

has the solution such that $A + RX$ stable when $X = x_2 x_1^{-1}$ where

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \qquad \text{has stable eigenvectors} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

H_2 (LQR):

$$H = \begin{bmatrix} A & -B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}$$

H_∞ (no eigenvalues on the imaginary axis and x_1 nonsingular):

$$H = \begin{bmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}$$

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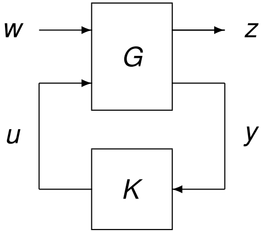
H_∞ synthesis

Conditions for the existence of an H_∞ controller with a closed-loop gain less than γ:



- $H_\infty \in \mathrm{dom}(\mathrm{Ric})$
- $X_\infty = \mathrm{Ric}\, H_\infty \succeq 0$
- $J_\infty \in \mathrm{dom}(\mathrm{Ric})$
- $Y_\infty = \mathrm{Ric}\, J_\infty \succeq 0$
- $\rho(X_\infty Y_\infty) < \gamma^2$
- $\gamma > \|D_{11} D_{21}^{\perp}\|$
- $\gamma > \|D_{12}^{T\perp T} D_{11}\|$

H_∞ synthesis

Assume that D_{12} is full column rank

$$H_\infty = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} - \begin{bmatrix} B_1 & B_2 & 0 \\ 0 & 0 & -C_1^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 & D_{11}^T \\ 0 & 0 & D_{12}^T \\ D_{11} & D_{12} & -I \end{bmatrix}^{-1} \begin{bmatrix} 0 & B_1^T \\ 0 & B_2^T \\ C_1 & 0 \end{bmatrix}$$

and assume that D_{21} is full row rank

$$J_\infty = \begin{bmatrix} A^T & 0 \\ 0 & -A \end{bmatrix} - \begin{bmatrix} C_1^T & C_2^T & 0 \\ 0 & 0 & -B_1 \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 & D_{11} \\ 0 & 0 & D_{21} \\ D_{11}^T & D_{21}^T & -I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_1 \\ 0 & C_2 \\ B_1^T & 0 \end{bmatrix}$$

If D_{12} or D_{21} lose rank more analysis is needed (generalized eigenvalue problems). The problem can be solvable or it can be badly formulated (gain goes to infinity as γ approaches its optimal value).

Coprime factorization

Coprime factorization:

$$G = \underbrace{\tilde{M}^{-1}\tilde{N}}_{\text{LCF}} = \underbrace{NM^{-1}}_{\text{RCF}}$$

where $\tilde{M}$ (M) and $\tilde{N}$ (N) are stable.

The factorization is normalized if

$$\tilde{M}(-s)^*\tilde{M}(s) + \tilde{N}(-s)^*\tilde{N}(s) = I$$

Parrott's theorem

$$\gamma = \min_X \left\| \begin{bmatrix} B & X \\ D & C \end{bmatrix} \right\| = \max \left\{ \left\| \begin{bmatrix} D & C \end{bmatrix} \right\|, \left\| \begin{bmatrix} B \\ D \end{bmatrix} \right\| \right\}$$

One such X:

$$X = \begin{bmatrix} 0 & B \end{bmatrix} \begin{bmatrix} -\gamma & D \\ D^T & -\gamma \end{bmatrix}^{\dagger} \begin{bmatrix} C \\ 0 \end{bmatrix}$$

or

$$X = \arg\min_X \operatorname{rank} \begin{bmatrix} & B & X \\ -\gamma & D & C \\ D^T & -\gamma & \end{bmatrix}$$

LMIs

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \qquad \text{where} \quad F_i = F_i^T.$$

The set of $F(x) \succ 0$ is convex.

Schur complement

$$\begin{bmatrix} S & G^T \\ G & R \end{bmatrix} \prec 0$$

is equivalent to $S - G^T R^{-1} G \prec 0$ and $R \prec 0$.

The elimination lemma

The problem

$$Q + UKV^T + VK^TU^T \prec 0.$$

has a solution K, if and only if

$$U_\perp^T Q U_\perp \prec 0, \quad \text{where } U_\perp = \operatorname{null} U^T$$

and

$$V_\perp^T Q V_\perp \prec 0, \quad \text{where } V_\perp = \operatorname{null} V^T$$

are both satisfied.

SVD and Bounded Real Lemma

The maximum singular value $\bar{\sigma}(D) = \|D\| < \gamma$ is equivalent to

$$\begin{bmatrix} -\gamma & D^T \\ D & -\gamma \end{bmatrix} \prec 0$$

$\|G\|_\infty < \gamma$ is equivalent to that there exists a $P = P^T \succ 0$ such that

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma & D^T \\ C & D & -\gamma \end{bmatrix} \prec 0$$

LMI H_∞ synthesis

Conditions for existence of a controller

$$\begin{bmatrix} \mathscr{N}_X & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma \end{array}\right] \begin{bmatrix} \mathscr{N}_X & 0 \\ 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} \mathscr{N}_Y & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -\gamma & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma \end{array}\right] \begin{bmatrix} \mathscr{N}_Y & 0 \\ 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$$

where

$$\begin{bmatrix} C_2 & D_{21} \end{bmatrix} \mathscr{N}_X = 0 \quad \text{and} \quad \begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix} \mathscr{N}_Y = 0$$

Design

Design methods:

SISO methods: LQR, PID, Lead-lag.

MIMO-metoder: H_∞, loop shaping, μ-synthesis.

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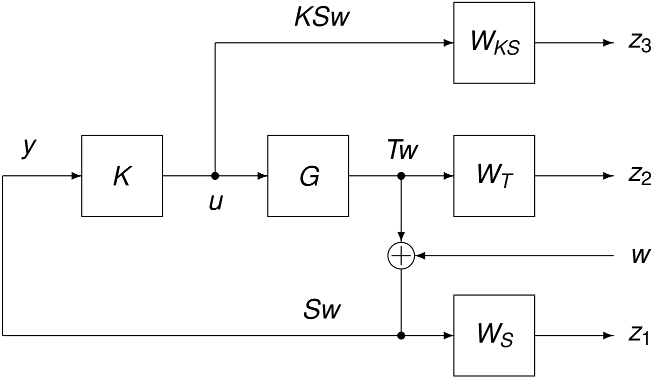
Design – standard structure in closed loop



$$KS \leq W_{KS}^{-1} \qquad T \leq W_T^{-1} \qquad S \leq W_S^{-1}$$

Structure

- Structure: What do I want to achieve?
- Standard structure
- MIMO: amplitude and phase margins at the input can be provided by adding extra inputs and outputs.

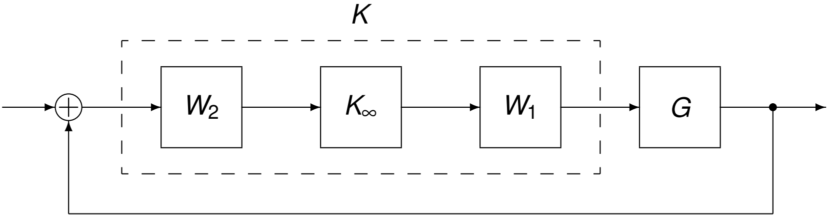
H_∞-design

- Everything must be specified, non-forgiving, nothing is for free.
- Sometimes a controller with too high bandwidth is obtained – try to relax the requirements and reduce the bandwidth.
- Full rank on D_{12} and D_{21} means direct terms in W_{KS} (and $w \to y$).
- Results in high-order controllers.

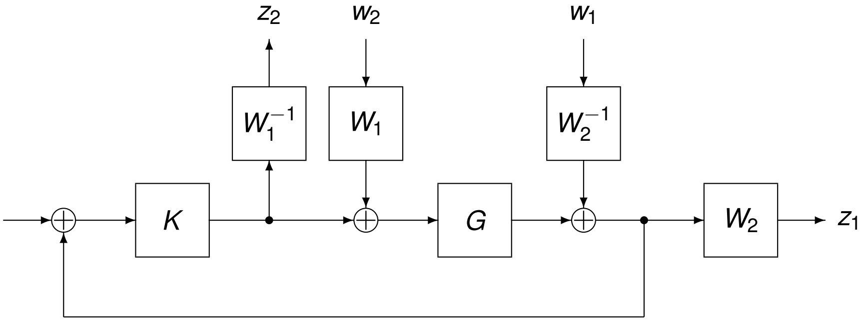
Loop shaping

Design W_1 and W_2 such that $W_1 G W_2$ satisfies the desired loop gain.

Use `ncfsyn` for synthesis ($\varepsilon \geq 0.2 - 0.3$).

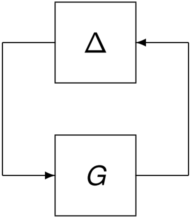


Alternative formulation



Small gain theorem

For analysis, the small gain theorem is an important tool.



G and Δ are stable.

$$\left\{ \begin{array}{l} \|\Delta\|_\infty \leq 1/\gamma \\ \|G\|_\infty < \gamma \end{array} \right. \Rightarrow \text{closed loop system is stable.}$$

Δ can be nonlinear with limited power gain (induced L_2-norm).

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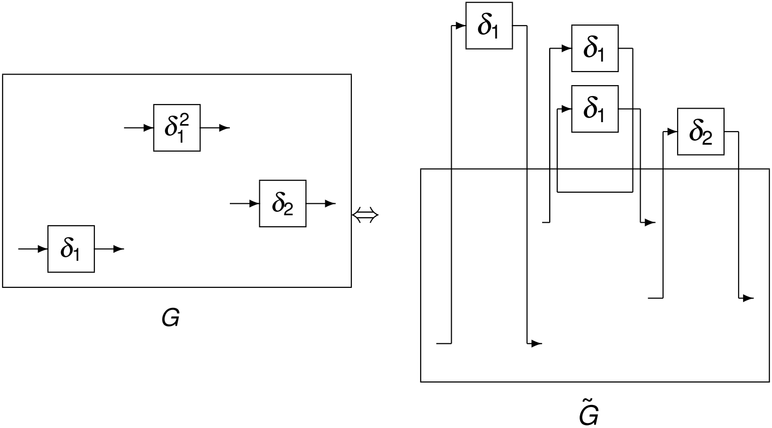
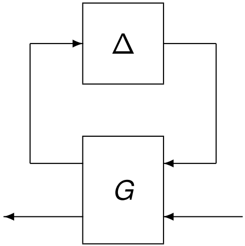
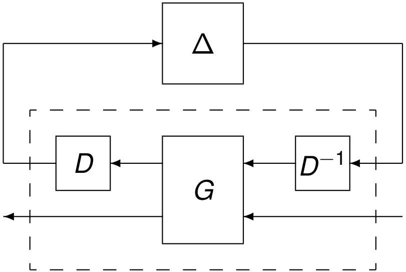
Pull out the parameters



Here we have repeated uncertainties, δ_1.

D-scalings: $D\Delta = \Delta D$, D invertible





μ-design

(i) Find a K such that $\|\mathscr{F}_\ell(G,K)\|_\infty$ is minimized (H_∞-synthesis).

(ii) Find a D (with fixed K) such that $\left\|D\mathscr{F}_\ell(G,K)D^{-1}\right\|_\infty$ is minimized.

(iii) Find a K (with fixed D) such that $\left\|D\mathscr{F}_\ell(G,K)D^{-1}\right\|_\infty$ is minimized (H_∞-synthesis).

(iv) Repeat (ii) and (iii) till convergence or till $\gamma = \left\|D\mathscr{F}_\ell(G,K)D^{-1}\right\|_\infty$ becomes sufficiently small.

Model reduction

H_∞-design often results in high-order controllers, sometimes 50 to 100 states (weights + G).

Unrealistic to implement.

Steps of reduction:

- Use balancing and truncation for removing states that are barely visible.
- Remove past poles (significantly higher than the bandwidth). Replace by a constant term.
- Use more advanced methods, such as weighted model reduction or `hinfstruct`.