1 A Flexible Ring Example

Consider a flexible ring controlled by a number of force actuators placed at even intervals around the ring. Position sensors are located at the same places as the actuators.

The dynamics of the flexible ring can be described as a mechanical resonant system. The normalized response from force input to position output is given by

$$G(s) = \sum_{i=0} \psi_i(\phi_y) \underbrace{\frac{1}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}}_{g_i(s)} \psi_i(\phi_u)$$

The modes are defined by their shapes, ψ_i, and their frequencies, ω_i. We assume that the dampings of the modes, ζ, are 0.01.

i	ψ_i	ω_i
0	$1/\sqrt{2}$	0
1	$\sin(\phi)$	0
2	$\cos(\phi)$	0
3	$\sin(2\phi)$	9
4	$\cos(2\phi)$	9
...	...	...
$2n-1$	$\sin(n\phi)$	$(n^2-1)^2$
$2n$	$\cos(n\phi)$	$(n^2-1)^2$
...	...	...

You can use `http://www.control.isy.liu.se/~andersh/teaching/genring.m` to generate this model.

```
m = 5;  % number of inputs and outputs
phi = (1:m)/m*2*pi;  % position of sensors and actuators
n = 5;  % 2*n+1 modes
[gs, om, psi, G] = genring (n, phi);
```

Try to design a controller with five inputs and five outputs ($m = p = 5$) to the plant such that

(i) the closed loop bandwidth should be at least 3 rad/s;

(ii) allow for a loop delay of at least 0.05 s before the closed loop becomes unstable.

(iii) the frequencies of the flexible modes should be allowed to vary ±20%.

(iv) allow for a margin of 10 dB for those modes that are not actively controlled ($\omega > 9$ rad/s).

1.1 Hints

Use the structure of the system: $G(s) = \Psi \operatorname{diag}(g_i(s)) \Psi^T$. For instance if you use a diagonal controller of the type $K(s) = k(s)I$, this structure can be employed when analyzing the closed loop system, since $K(s)$ commutes with Ψ or any other constant matrix.

Consider the loop gain of the system:

$$K(s)G(s) = k(s)\Psi\text{diag}(g_i(s))\Psi^T$$

or

$$\Psi^T\Psi\text{diag}(k(s)g_i(s))$$

if we assume that all controllers $k(s)$ are equal.

First note that $\Psi^T\Psi$ is structured. For the nominal system we get

$$\Psi^T\Psi = \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This structure reveals that the modes are coupled together, here in pairs. The exception is mode 10, which is neither observable nor controllable. Instead mode 9 gets twice the gain.

Table 1: Tranfer function of the system where sensors and actuators are at the same positions.

i	ω_i	transfer function
0, 9	0, 576	$\frac{5}{2}(g_0(s) + 2g_9(s))$
1, 7	0, 255	$\frac{5}{2}(g_1(s) + g_7(s))$
2, 8	0, 255	$\frac{5}{2}(g_2(s) + g_8(s))$
3, 5	9, 64	$\frac{5}{2}(g_3(s) + g_5(s))$
4, 6	9, 64	$\frac{5}{2}(g_4(s) + g_6(s))$
10	576	$0 \times g_{10}(s)$

The controller, $k(s)$, that we search for should be able to control any of these six combined systems. Note that in the nominal configuration, there are essentially only three different systems: $g_0(s) + 2g_9(s)$, $g_1(s) + g_7(s)$, and $g_3(s) + g_5(s)$ (not counting the last zero system).

The first three modes are essentially double integrators: $2.5/s^2$. We include a delay of 0.1 s in order to allocate for some extra phase margins. We also add a low pass filter in order to avoid excitation of the 64 rad/s modes.

```
[num, den] = pade (0.1, 1); del = tf(num,den);
w1 = tf(1,[1/15 1]);
[k1, cl, gam] = ncfsyn (g0*del, 10*w1);
disp(gam)
    3.9923
```

The gain margins become 13.4, 13.5, and 11.4 dB, while the phase margins become 54, 54, and 36 deg. The closed-loop system becomes unstable at a delay of 0.058 s.

This is just a first iteration of a controller. Try to improve the design yourself by shaping the filters in a better way to reflect the performance specification. Check your design by bode plots and step responses.

What can you say about the response of the full 5×5 system?

1.2 Modified problem

One of the difficulties lies in $g_3(s)+g_5(s)$ and the similar $g_4(s)+g_6(s)$. These two combined systems have modes in 9 and 64 rad/s. If we would like to control g_3 actively, but still keep the bandwidth below say 20 rad/s, the gain

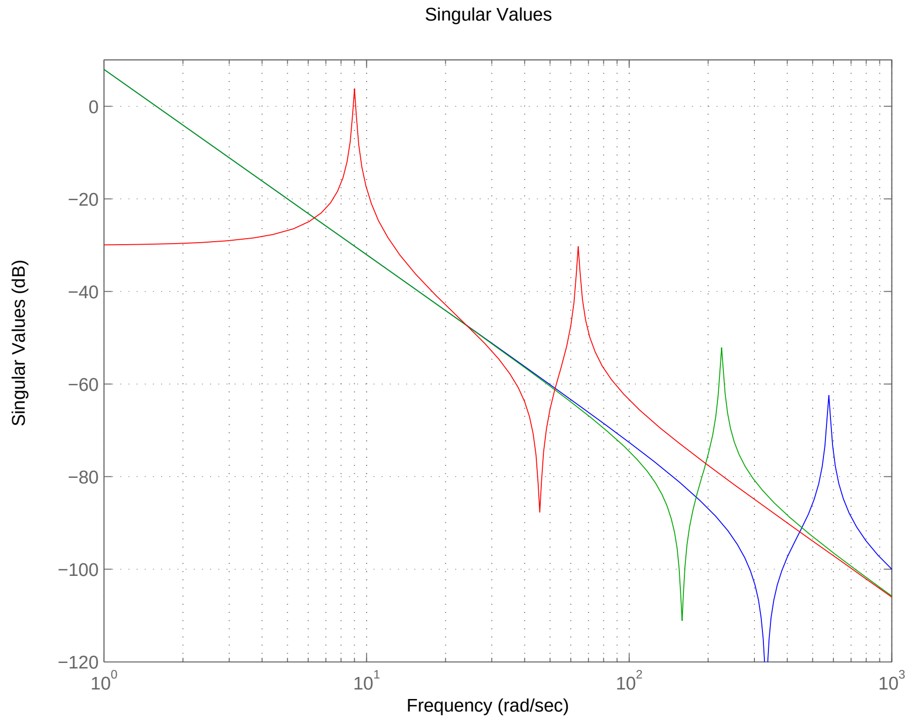

Figure 1: Tranfer function of the system where sensors and actuators are at the same positions.

Table 2: Tranfer function of the system where sensors are place in between the actuators.

i	ω_i	transfer function
0, 9	0, 576	$2.5(g_0(s) - 2g_9(s))$
1, 7	0, 255	$2.5\cos(\frac{\pi}{5})(g_1(s) - g_7(s))$
2, 8	0, 255	$2.5\cos(\frac{\pi}{5})(g_2(s) - g_8(s))$
3, 5	9, 64	$2.5\cos(\frac{2\pi}{5})(g_3(s) - g_5(s))$
4, 6	9, 64	$2.5\cos(\frac{2\pi}{5})(g_4(s) - g_6(s))$
10	576	$0 \times g_{10}(s)$

at g_5 must be reduced using a low-pass filter. This will induce a phase-loss at lower frequencies, which means that the phase margin will be reduced.

In order to improve the situation we place the actuators in the middle between the sensors and as input to each actuator we use the average between the two closest sensors.

```
[gs, om, psi2, G] = genring(n,phi+1/m*pi);
T = 0.5*toeplitz([1 1 zeros(1,m-2)], [1 zeros(1,m-1)]); T(1,m) = 0.5;
psi'*T*psi2

ans =

    2.5000         0   -0.0000    0.0000   -0.0000    0.0000    0.0000   -0.0000    0.0000   -3.5355    0.0000
   -0.0000    2.0225   -0.0000    0.0000   -0.0000    0.0000    0.0000   -2.0225    0.0000    0.0000    0.0000
         0   -0.0000    2.0225    0.0000         0    0.0000    0.0000    0.0000    2.0225    0.0000    0.0000
   -0.0000   -0.0000   -0.0000    0.7725    0.0000   -0.7725    0.0000    0.0000    0.0000    0.0000    0.0000
   -0.0000   -0.0000   -0.0000    0.0000    0.7725    0.0000    0.7725    0.0000   -0.0000    0.0000    0.0000
    0.0000    0.0000    0.0000    0.7725   -0.0000   -0.7725    0.0000   -0.0000    0.0000   -0.0000    0.0000
   -0.0000   -0.0000   -0.0000   -0.0000   -0.7725   -0.0000   -0.7725    0.0000   -0.0000    0.0000   -0.0000
    0.0000    2.0225   -0.0000   -0.0000   -0.0000    0.0000    0.0000   -2.0225    0.0000   -0.0000    0.0000
   -0.0000   -0.0000   -2.0225   -0.0000    0.0000   -0.0000   -0.0000    0.0000   -2.0225    0.0000   -0.0000
    3.5355         0   -0.0000    0.0000   -0.0000    0.0000    0.0000   -0.0000    0.0000   -5.0000    0.0000
   -0.0000   -0.0000    0.0000   -0.0000    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000   -0.0000
```

$$\Psi^T T \Psi_2 = \begin{bmatrix} 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.5355 & 0 \\ 0 & 2.0225 & 0 & 0 & 0 & 0 & 0 & 2.0225 & 0 & 0 & 0 \\ 0 & 0 & 2.0225 & 0 & 0 & 0 & 0 & 0 & 2.0225 & 0 & 0 \\ 0 & 0 & 0 & 0.7725 & 0 & 0.7725 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7725 & 0 & 0.7725 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7725 & 0 & -0.7725 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.7725 & 0 & -0.7725 & 0 & 0 & 0 & 0 \\ 0 & 2.0225 & 0 & 0 & 0 & 0 & 0 & -2.0225 & 0 & 0 & 0 \\ 0 & 0 & -2.0225 & 0 & 0 & 0 & 0 & 0 & -2.0225 & 0 & 0 \\ 3.5355 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

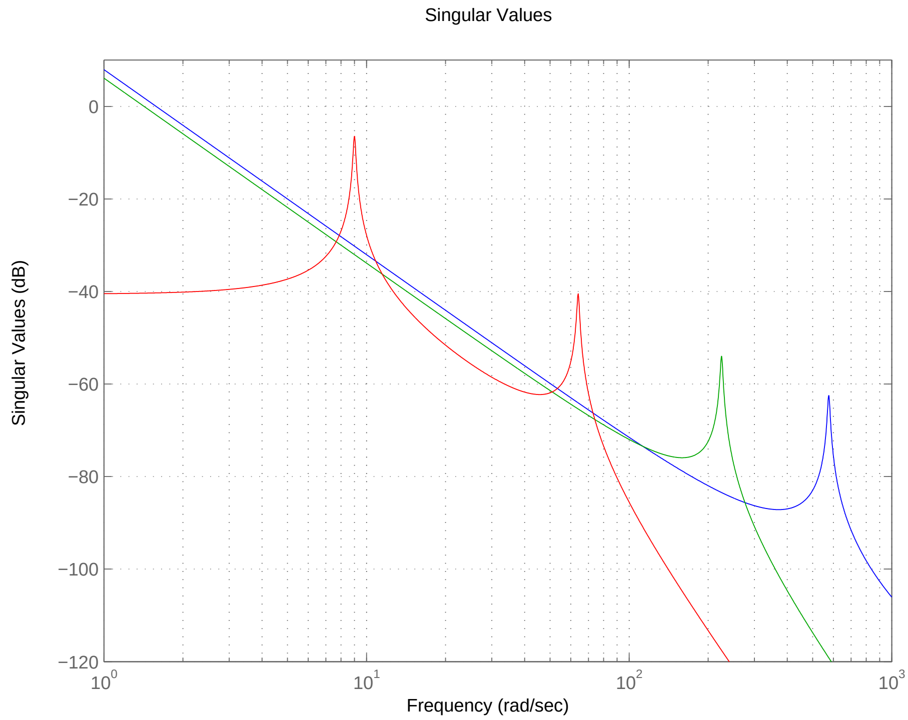

Figure 2: Tranfer function of the system where sensors are place in between the actuators. The resonance at 64 rad/s is now down to less than -40 dB, which is about 10 dB lower compared to the colocated sensor-actuator case.