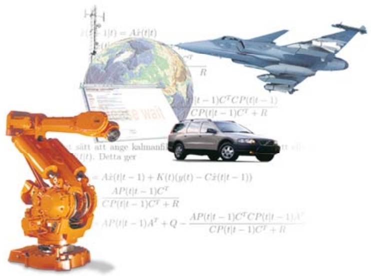
Digital Signal Processing, Lecture 9
Kalman Filter – Derivation



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Outline Lecture 9

1. Summary of lecture 8
2. Problem formulation
3. Useful notation
4. Structure of the Kalman filter
5. Derivation of the Kalman filter
   a) Time update
   b) Measurement update

Summary of Lecture 8 (I/II)

FIR Wiener filter – solution provided by a finite number of linear equations,

$$\sum_{i=0}^{m-1} h(i) R_{yy}(k-i) = R_{sy}(k), \qquad k = 0, 1, \ldots, m-1$$

General causal Wiener filter results in infinitely many equations.

Trick: Use a whitening filter

Summary of Lecture 8 (I/II)

Resulting Wiener filtering algorithm (Alg. 7.1)

Select m > 0 for prediction, m < 0 for smoothing and m = 0 for filtering.

1. Spectral factorization

$$\Phi_{yy}(z) = \sigma_e^2 T(z) T(1/z)$$

2. Compute the filter (let the direct term be part of the causal term)

$$H_e(z) = \frac{z^m \Phi_{sy}(z)}{\sigma_e^2 T(1/z)} = \underbrace{\sum_{k=0}^{\infty} h_e(k) z^{-k}}_{[H_e(z)]_+} + \underbrace{\sum_{k=-\infty}^{-1} h_e(k) z^{-k}}_{[H_e(z)]_-}$$

3. The Wiener filter is now given by $\hat{s}(t) = H(z) y(t)$, where

$$H(z) = \frac{[H_e(z)]_+}{z^m T(z)}$$

Kalman Filter – Signal Model

Signal model

$$x(t+1) = Ax(t) + w(t),$$
$$y(t) = Cx(t) + v(t),$$

where w(t) and v(t) are white noise, with

$$\mathsf{E}(w(t)) = \mathsf{E}(v(t)) = 0,$$
$$\mathsf{Cov}(w(t)) = \mathsf{E}(w(t)w^T(t)) = Q,$$
$$\mathsf{Cov}(v(t)) = \mathsf{E}(v(t)v^T(t)) = R,$$

$\mathsf{E}(w(t)v^T(t)) = S$. (We will assume that S = 0 for simplicity)

The initial state x(0) is also assumed to be a stochastic variable with

$$\mathsf{E}(x(0)) = x_0, \qquad \mathsf{Cov}(x(0)) = \Pi_0$$

Typically, all stochastic variables are assumed Gaussian.

Markov Property of the State-Space Model

Fundamental property of the state variable:

The state x(t) contains all information we have about the system at time t.

That is, given x(t), there is no useful information available in the previous measurements and the previous states.

This property is known as the Markov property.

Three Basic Problems

There are **three** problems that are relevant for optimal estimation.

Computing an estimate of x(t) using the information

$$Y^s = \{y(1), y(2), \ldots, y(s)\}$$

results in the following three problems

1. If t = s, **filtering** ← We will look at the filtering problem today
2. If t > s, **prediction**
3. If t < s, **smoothing**

Kalman Filter – Problem Definition

Problem: Compute an estimate of the state vector x(t) for a state-space model

$$x(t+1) = Ax(t) + w(t),$$
$$y(t) = Cx(t) + v(t),$$

using the information available in all the previous measurements

$$y(\tau), \qquad 0 \leq \tau \leq t$$

such that the covariance of the estimation error

$$\tilde{x}(t \mid t) = x(t) - \hat{x}(t \mid t)$$

is minimized.

The Kalman Filter

Time update

$$\hat{x}(t+1|t) = A\hat{x}(t|t),$$
$$P(t+1|t) = AP(t|t)A^T \underbrace{+Q}_{\text{Increase uncertainty}}.$$

Measurement update

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K(t)\overbrace{(y(t) - C\hat{x}(t|t-1))}^{\text{Innovation}},$$
$$L(t) = CP(t|t-1)C^T + R,$$
$$K(t) = P(t|t-1)C^T L(t)^{-1},$$
$$P(t|t) = P(t|t-1) \underbrace{-P(t|t-1)C^T L(t)^{-1} C P(t|t-1)}_{\text{Decrease uncertainty}},$$

A Few Concepts to Summarize Lecture 8

Dynamic equation: Explains the dynamics of the system under study, that is how does the state vector evolve over time.

Measurement equation: Explains how the measurement is related to the state vector. Models the sensors.

Innovation: The new information available in a measurement.

Time update: The part of the Kalman filter that makes use of the dynamic equation in order to predict the state to the next time step.

Measurement update: The part of the Kalman filter that explains how the information available in the new measurement is incorporated into the estimate.