Summary Lecture 2

<table>
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<th>Def.</th>
<th>TT1</th>
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<th>MCT, UT</th>
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<tr>
<td>$y = h(x) + e$</td>
<td>NLS $\Rightarrow \hat{x}, P$</td>
<td>WLS</td>
<td>WLS with comp., Indirect approach: Gaussian Lemma</td>
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<tr>
<td>$x = h^{-1}(y - e)$</td>
<td>Inv. Map $\Rightarrow x_k$</td>
<td>i) Estimate $x_k, P_k$</td>
<td>ii) Use SF formula $\Rightarrow \hat{x}, P$</td>
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Gaussian Lemma:
Assume

$$u = \begin{pmatrix} x \\ e \end{pmatrix} \sim \mathcal{N} \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix}, \begin{pmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{pmatrix}$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ h(x, e) \end{pmatrix} \sim \mathcal{N} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}, \begin{pmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{pmatrix}$$

Then

$$\hat{x} = \hat{x} + P_{yy}(P_{yy})^{-1}(y - \hat{y})$$

$$\text{cov}(\hat{x}) = P_{xx} - P_{yy}(P_{yy})^{-1}P_{yx}.$$
Cramér-Rao Lower Bound (CRLB)

Theorem (Cramér-Rao Lower Bound)

For any unbiased \( \hat{x} \) (\( E(\hat{x}) = x^0 \)), the estimate satisfies

\[
\text{cov}(\hat{x}) \preceq I^{-1}(x^0),
\]

where \( I(x) \) is the Fisher information matrix (FIM),

\[
I(x) = -E\left\{ \nabla_x^2 \log p_e(y - h(x)) \right\}
= E\left\{ \nabla^T_x \left( \log p_e(y - h(x)) \right) \nabla_x \left( \log p_e(y - h(x)) \right) \right\}
\]

under mild regularity conditions, see Appendix C

Sensor Models in Radio Networks

Received signal \( y_k(t) \) as a noisy, delayed and attenuated version of the transmitted signal \( s(t) \)

\[
y_k(t) = a_k s(t - \tau_k) + e_k(t), \quad k = 1, 2, \ldots, N
\]

Time-delay estimation using known training signal (pilot symbols) gives

\[
r_k = \tau_k + \|x - p_k\|_2 = \sqrt{(x_1 - p_{k,1})^2 + (x_2 - p_{k,2})^2}
\]

- **Time-of-arrival (TOA)** — transport delay
- **Time-difference-of-arrival (TDOA)** — arrival time known, but the broadcast time not
- Estimation of \( a_k \) gives received signal strength (RSS), which does not require known training signal, just transmitter power \( P_0 \) and path propagation constant \( \alpha \)

\[
P_k = P_0 - \alpha \log(\|x - p_k\|)
\]

Basic Network Sensor Models

The basic network measurements in any network (radio, acoustic, sonar, seismic) are summarized as follows:

- **TOA** \( h_k(x) = r_k = \|x - p_k\| \)
- **TDOA** \( h_k(x) = r_k = \|x - p_k\| + r_0 \)
- **DOA** \( h_k(x) = \phi_k = \arctan2(x_2 - p_{k,2}, x_1 - p_{k,1}) \)
- **RSS** \( h_k(x) = P_0 - \beta \log(\|x - p_k\|) \)

Note:

All models are on the form \( y_k = h_k(x) + e_k \)
TDOA Geometry

Common offset \( r_0 \) (due to unsynchronized clocks)

\[ r_k = \|x - p_k\| + r_0, \quad k = 1, 2, \ldots, N \]

Estimation approach:
Consider \( r_0 \) as a parameter (cf. GPS)

Common in literature:
Study range differences

\[ r_{i,j} = r_i - r_j, \quad 1 \leq i < j \leq N \]

Gives nice geometric interpretation

TDOA: maths

Assume \( p_1 = (D/2, 0)^T \) and \( p_2 = (-D/2, 0)^T \), respectively, then

\[ r_1 = \sqrt{x_2^2 + (x_1 - D/2)^2} \]
\[ r_2 = \sqrt{x_2^2 + (x_1 + D/2)^2} \]
\[ r_{12} = r_2 - r_1 = h(x, D) = \sqrt{x_2^2 + (x_1 + D/2)^2} - \sqrt{x_2^2 + (x_1 - D/2)^2} \]

Simplify

\[ \frac{x_1^2}{a} - \frac{x_2^2}{b} = \frac{x_1^2}{r_{12}^2/4} - \frac{x_2^2}{D^2/4 - r_{12}^2/4} = 1. \]

Direction/Angle of Arrival (DOA/AOA)

The solution to this hyperbolic equation has asymptotes along the lines

\[ x_2 = \pm \frac{b}{a} x_1 = \pm \sqrt{\frac{D^2}{4} - \frac{r_{12}^2}{4}} \]
\[ x_1 = \pm x_1 \sqrt{\left(\frac{D}{n_2}\right)^2 - 1} \]

AOA, \( \varphi \), for far-away transmitters (the far-field assumptions of planar waves):

\[ \varphi = \pm \arctan \left( \sqrt{\left(\frac{D}{n_2}\right)^2 - 1} \right) \]
### Estimation Criteria

General problem formulation and solution:

\[ \hat{x} = \arg\min_x V(x). \]

Specific cases summarized:

- **NLS**  \[ V^{\text{NLS}}(x) = \|y - h(x)\|^2 = (y - h(x))^T (y - h(x)) \]
- **WNLS**  \[ V^{\text{WNLS}}(x) = (y - h(x))^T R^{-1}(x) (y - h(x)) \]
- **ML**  \[ V^{\text{ML}}(x) = \log p_e (y - h(x)) \]
- **GML**  \[ V^{\text{GML}}(x) = (y - h(x))^T R^{-1}(x) (y - h(x)) + \log \det R(x) \]

### Estimation Methods

**General principles:**

**Steepest descent (Stochastic gradient)**  \[ \hat{x}_k = \hat{x}_{k-1} + \mu_k H^T (\hat{x}_{k-1} R^{-1}(y - h(\hat{x}_{k-1})) \]

**Gauss-Newton**  \[ \hat{x}_k = \hat{x}_{k-1} + \mu_k (H^T (\hat{x}_{k-1} R^{-1} H(\hat{x}_{k-1})))^{-1} H^T (\hat{x}_{k-1} R^{-1}(y - h(\hat{x}_{k-1})) \]

**Problem specific quantities:**

<table>
<thead>
<tr>
<th>Method</th>
<th>( h(x, p_i) )</th>
<th>( \partial h / \partial x_1 )</th>
<th>( \partial h / \partial x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSS</td>
<td>( P_0 + 10 \beta \log_{10} r_i )</td>
<td>( 10 \beta \frac{x_1 - p_{1,i}}{x_1 - p_{1,1}} )</td>
<td>( 10 \beta \frac{x_2 - p_{2,2}}{r_i^2} )</td>
</tr>
<tr>
<td>TOA</td>
<td>( r_i )</td>
<td>( r_i \frac{x_1 - p_{1,1}}{x_1 - p_{1,1}} )</td>
<td>( 10 \log_{10} \frac{x_2 - p_{2,2}}{r_i^2} )</td>
</tr>
<tr>
<td>TDOA</td>
<td>( r_i - r_j )</td>
<td>( \frac{x_1 - p_{1,1}}{D_j} - \frac{x_1 - p_{1,2}}{D_j} )</td>
<td>( \frac{x_2 - p_{2,2}}{D_j} - \frac{x_2 - p_{2,2}}{D_j} )</td>
</tr>
<tr>
<td>AOA</td>
<td>( \alpha_i + \arctan \frac{x_2 - p_{2,2}}{x_i - p_{1,1}} )</td>
<td>( \frac{x_1 - p_{1,1}}{D_j} - \frac{x_1 - p_{1,2}}{D_j} )</td>
<td>( \frac{x_2 - p_{2,2}}{D_j} - \frac{x_2 - p_{2,2}}{D_j} )</td>
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</tbody>
</table>
The steepest descent and Gauss-Newton algorithms for TOA.

CRLB for TOA and TDOA: \( \text{RMSE} \left( \hat{x} \right) \geq \sqrt{\text{tr}(I^{-1}(x^0))} \), where \( I^{-1}(x^0) \) is the Fisher information matrix (FIM) evaluated at the true parameter \( x^0 \).

Dedicated Explicit LS Solutions

Basic trick: study NLS of squared distance measurements:

\[
\hat{x} = \arg \min_x \sum_{k=1}^N (r_k^2 - \|x - p_k\|^2)^2.
\]

Note:
What does this imply for the measurement noise? Several ad hoc solutions exist for transforming the nonlinear problem into a linear problem.
DOA Triangulation

Angle observations from sensor at position $p_k$

\[
\phi_k = \arctan \left( \frac{x_2 - p_{k,2}}{x_1 - p_{k,1}} \right)
\]

\[
(x_1 - p_{k,1}) \tan(\phi_k) = x_2 - p_{k,2}
\]

Linear model immediate

\[
y = h x + e
\]

\[
y = \begin{pmatrix}
\rho_{1,1} \tan(\phi_1) - p_{1,2} \\
\rho_{2,1} \tan(\phi_2) - p_{2,2} \\
\vdots \\
\rho_{N,1} \tan(\phi_N) - p_{N,2}
\end{pmatrix},
\quad
h = \begin{pmatrix}
\tan(\phi_1) & -1 \\
\tan(\phi_2) & -1 \\
\vdots & \vdots \\
\tan(\phi_N) & -1
\end{pmatrix}
\]

RSS (1/2)

Received signal strength (RSS) observations:

- All waves (radio, radar, IR, seismic, acoustic, magnetic) decay exponentially in range
- Receiver $k$ measures energy/power/signal strength for wave $i$:
  \[
P_{k,i} = P_{0,i} \|x - p_k\|^\beta_i
\]

- Transmitted signal strength and path loss constant unknown
- Communication constraints make coherent detection from the signal waveform impossible
- Compare $P_{k,i}$ for different receivers

RSS (2/2)

Log model:

\[
\bar{P}_{k,i} = \bar{P}_{0,i} + \beta_i \log(\|x - p_k\|)
\]

\[
y_{k,i} = \bar{P}_{k,i} + e_{k,i}
\]

Use separable least squares to eliminate path loss constant and transmitted power for wave $i$:

\[
(x, \theta) = \arg \min_{x, \theta} V(x, \theta)
\]

\[
V(x, \theta) = \sum_{i=1}^{M} \sum_{k=1}^{N} \frac{(y_{k,i} - h(c_k(x), \theta_i))^2}{\sigma_{p,i}^2}
\]

\[
h(c_k(x), \theta_i) = \theta_{i,1} + \theta_{i,2} c_k(x)
\]

\[
c_k(x) = \log(\|x - p_k\|)
\]

Finally, use NLS to optimize over 2D target position $x$.

Shooter Localization (1/2)

Network with 10 microphones around a ‘camp’. Shooter aiming inside the camp. Supersonic bullet injects a shock wave followed by an acoustic muzzle blast. Fusion of time differences give shooter position and aiming point.

Model, with $x$ being the position of the shooter,

\[
x_k^{sw} = t_0 + b_k + \frac{1}{2} r \log(\frac{v_0}{v_0 - r \|d_k - x\|}) + e_k^{sw}
\]

where $d_k$ is an implicit function of the state.
Shooter Localization: results

- WLS loss function illustrates the information in $y_{kMB} - y_{kSW}$, $k = 1, 2, \ldots, N = 10$
- Both shooter position and aiming direction $\alpha$ are well estimated for each shot.
- Also bullet’s muzzle speed and ammunition length can be estimated!

Summary Lecture 3 (1/2)

- Signal model for localization in sensor networks
  $$y = h(x - p) + e, \quad \text{cov}(e) = R$$
  
  $x$ is the unknown position, $p$ is the (known in this chapter) sensor locations.
- The basic network measurements:
  - TOA $r_k = \|x - p_k\| + e_k$
  - TDOA $r_k = \|x - p_k\| + r_0 + e_k$
  - DOA $\phi_k = \arctan2(x_2 - p_{k,2}, x_1 - p_{k,1}) + e_k$
  - RSS $y_k = P^0 - \beta \log(\|x - p_k\|)$
- NLS or NLT general approaches to estimate $x$

Summary Lecture 3 (2/2)

- Tricks (not statistically optimal!)
  - TOA range parameter trilateration: $r_k^2$ is linear in $(x_1, x_2, x_1^2 + x_2^2)^T$
  - TOA reference sensor trilateration: $r_k^2 - r_1^2$ is linear in $x$
  - DOA triangulation approach: $x$ is an affine function in $\tan(\phi_k)$