TSRT14: Sensor Fusion
Lecture 3
Sensor networks
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Whiteboard:
- Selected derivations.

Frames:
- Typical sensor networks measurements
- Nonlinear models in sensor networks
- Dedicated LS solutions.
### Summary Lecture 2

<table>
<thead>
<tr>
<th>Def.</th>
<th>TT1</th>
<th>TT2</th>
<th>MCT, UT</th>
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<tr>
<td>(y = h(x) + e)</td>
<td>NLS (\Rightarrow \hat{x}, P)</td>
<td>WLS</td>
<td>WLS with comp.</td>
</tr>
<tr>
<td>(x = h^{-1}(y - e))</td>
<td>Inv. Map (\Rightarrow x_k)</td>
<td>i) Estimate (x_k, P_k)</td>
<td>ii) Use SF formula (\Rightarrow \hat{x}, P)</td>
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</table>

**Gaussian Lemma:**

Assume

\[
\begin{align*}
    u &= \begin{pmatrix} x \\ e \end{pmatrix} \in \mathcal{N} \left( \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}, \begin{pmatrix} P^{xx} & 0 \\ 0 & R \end{pmatrix} \right) \\
    z &= \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ h(x, e) \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} P^{xx} & P^{xy} \\ P^{yx} & P^{yy} \end{pmatrix} \right)
\end{align*}
\]

Then

\[
\hat{x} = \bar{x} + P^{xy}(P^{yy})^{-1}(y - \bar{y}),
\]

\[
\text{cov}(\hat{x}) = P^{xx} - P^{xy}(P^{yy})^{-1}P^{yx}.
\]
Chapter 4 overview

- Sensor model $y_k = h_k(x) + e_k$ as before
- Basic sensor models, specific forms of $h(x)$.
- Geometrical interpretation (TDOA)
- Overview of algorithms with examples
- Tricks: transform to linear model or conditionally linear model
Theorem (Cramér-Rao Lower Bound)

For any unbiased $\hat{x}$ ($E(\hat{x}) = x^0$), the estimate satisfies

$$\text{cov}(\hat{x}) \succeq I^{-1}(x^0),$$

where $I(x)$ is the Fisher information matrix (FIM),

$$I(x) = -E\left(\nabla_x^2 \log p_e(y - h(x))\right),$$

$$= E\left\{ \nabla_x^T \left( \log p_e(y - h(x)) \right) \nabla_x \left( \log p_e(y - h(x)) \right) \right\}$$

under mild regularity conditions, see Appendix C
Cramér-Rao Lower Bound (CRLB): theorems

Theorems:

- \( \hat{x}^{ML} \) is efficient: asymptotically \( \mathbb{E}(\hat{x}) = x^0 \) and \( \text{cov}(\hat{x}) = \mathcal{I}^{-1}(x^0) \)

- If \( h(x) = Hx \), then:
  - \( \hat{x}^{WLS} \) is a \textit{best linear unbiased estimator} (BLUE)
  - \( \text{cov}(\hat{x}^{WLS}) = (H^T R^{-1} H)^{-1} \succeq \mathcal{I}^{-1}(x^0) \)
    with equality if and only if \( \mathbf{e} \) is Gaussian

- If \( p_e(x) = \mathcal{N}(\mathbf{e}; \mathbf{0}, R) \), then
  \[
  \mathcal{I}(x) = (\nabla_x^T h(x)) R^{-1} (\nabla_x h(x))
  \]
Sensor Models in Radio Networks

Received signal \( y_k(t) \) as a noisy, delayed and attenuated version of the transmitted signal \( s(t) \)

\[
y_k(t) = a_k s(t - \tau_k) + e_k(t), \quad k = 1, 2, \ldots, N
\]

Time-delay estimation using known training signal (pilot symbols) gives

\[
r_k = \tau_k v = \| x - p_k \|_2 = \sqrt{(x_1 - p_{k,1})^2 + (x_2 - p_{k,2})^2}
\]

- **Time-of-arrival (TOA)** — transport delay
- **Time-difference-of-arrival (TDOA)** — arrival time known, but the broadcast time not

Estimation of \( a_k \) gives received signal strength (RSS), which does not require known training signal, just transmitter power \( P_0 \) and path propagation constant \( \alpha \)

\[
P_k = P_0 - \alpha \log(\| x - p_k \|)
\]
Basic Network Sensor Models

The basic network measurements in any network (radio, acoustic, sonar, seismic) are summarized as follows:

- **TOA** \( h_k(x) = r_k = \| x - p_k \| \),
- **TDOA** \( h_k(x) = r_k = \| x - p_k \| + r_0 \),
- **DOA** \( h_k(x) = \varphi_k = \arctan2 \left( x_2 - p_{k,2}, x_1 - p_{k,1} \right) \), and
- **RSS** \( h_k(x) = P^0 - \beta \log \left( \| x - p_k \| \right) \)

**Note:**

All models are on the form \( y_k = h_k(x) + e_k \)
Common offset \( r_0 \) (due to unsynchronized clocks)

\[
r_k = \|x - p_k\| + r_0, \quad k = 1, 2, \ldots, N
\]

**Estimation approach:**
Consider \( r_0 \) as a parameter (cf. GPS)

**Common in literature:**
Study range differences

\[
r_{i,j} = r_i - r_j, \quad 1 \leq i < j \leq N
\]

Gives nice geometric interpretation
Assume \( p_1 = (D/2, 0)^T \) and \( p_2 = (-D/2, 0)^T \), respectively, then

\[
\begin{align*}
    r_1 &= \sqrt{x_2^2 + (x_1 - D/2)^2} \\
    r_2 &= \sqrt{x_2^2 + (x_1 + D/2)^2} \\
    r_{12} &= r_2 - r_1 = h(x, D) \\
    &= \sqrt{x_2^2 + (x_1 + D/2)^2} - \sqrt{x_2^2 + (x_1 - D/2)^2}
\end{align*}
\]

Simplify

\[
\frac{x_1^2}{a} - \frac{x_2^2}{b} = \frac{x_1^2}{r_{12}^2/4} - \frac{x_2^2}{D^2/4 - r_{12}^2/4} = 1.
\]
Illustration of three different values of \( r_{12} \). Corresponds to three different hyperbolic functions. Noise on \( r_{12} \) gives confidence bands.
The solution to this hyperbolic equation has asymptotes along the lines

$$x_2 = \pm \frac{b}{a} x_1 = \pm \sqrt{\frac{D^2/4 - r_{12}^2/4}{r_{12}^2/4}} x_1 = \pm x_1 \sqrt{\left(\frac{D}{r_{12}}\right)^2 - 1}$$

AOA, $\varphi$, for far-away transmitters (the far-field assumptions of planar waves):

$$\varphi = \pm \arctan\left(\sqrt{\left(\frac{D}{r_{12}}\right)^2 - 1}\right)$$
Noise-free nonlinear relations for TOA and TDOA.
Estimation Criteria

General problem formulation and solution:

\[ \hat{x} = \arg \min_x V(x). \]

Specific cases summarized:

- **NLS** \( V^{NLS}(x) = \|y - h(x)\|^2 = (y - h(x))^T(y - h(x)) \)
- **WNLS** \( V^{WNLS}(x) = (y - h(x))^T R^{-1}(x)(y - h(x)) \)
- **ML** \( V^{ML}(x) = \log p_e(y - h(x)) \)
- **GML** \( V^{GML}(x) = (y - h(x))^T R^{-1}(x)(y - h(x)) + \log \det R(x) \)
THE Example

Level curves $V(x)$ for TOA and TDOA.

Least squares loss function for TOA

Least squares loss function for TDOA

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General principles:

Steepest descent \((Stochastic\ gradient)\)

\[
\hat{x}_k = \hat{x}_{k-1} + \mu_k H^T(\hat{x}_{k-1})R^{-1}(y - h(\hat{x}_{k-1}))
\]

Gauss-Newton

\[
\hat{x}_k = \hat{x}_{k-1} + \mu_k \left(H^T(\hat{x}_{k-1})R^{-1}H(\hat{x}_{k-1})\right)^{-1}H^T(\hat{x}_{k-1})R^{-1}(y - h(\hat{x}_{k-1}))
\]

Problem specific quantities:

<table>
<thead>
<tr>
<th>Method</th>
<th>(h(x, p_i))</th>
<th>(\frac{\partial h}{\partial x_1})</th>
<th>(\frac{\partial h}{\partial x_2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSS</td>
<td>(P_0 + 10^\beta \log_{10} r_i)</td>
<td>(\frac{10^\beta}{\log 10} \frac{x_1-p_{i,1}}{r_i^2})</td>
<td>(\frac{10^\beta}{\log 10} \frac{x_2-p_{i,2}}{r_i^2})</td>
</tr>
<tr>
<td>TOA</td>
<td>(r_i)</td>
<td>(\frac{r_i}{x_1-p_{i,1}})</td>
<td>(\frac{r_i}{x_2-p_{i,2}})</td>
</tr>
<tr>
<td>TDOA</td>
<td>(r_i - r_j)</td>
<td>(\frac{x_1-p_{i,1}}{D_i} - \frac{x_1-p_{j,1}}{D_j})</td>
<td>(\frac{x_2-p_{i,2}}{D_i} - \frac{x_2-p_{j,2}}{D_j})</td>
</tr>
<tr>
<td>AOA</td>
<td>(\alpha_i + \arctan \frac{x_2-p_{i,2}}{x_1-p_{i,1}})</td>
<td>(-\frac{(x_1-p_{i,1})}{r_i^2})</td>
<td>(\frac{x_2-p_{i,2}}{r_i^2})</td>
</tr>
</tbody>
</table>
The steepest descent and Gauss-Newton algorithms for TOA.
The steepest descent and Gauss-Newton algorithms for TDOA.
CRLB for TOA and TDOA: \( \text{RMSE} (\hat{x}) \geq \sqrt{\text{tr}(I^{-1}(x^0))} \), where \( I^{-1}(x^0) \) is the *Fisher information matrix* (FIM) evaluated at the true parameter \( x^0 \).
Basic trick: study NLS of squared distance measurements:

\[ \hat{x} = \arg \min_x \sum_{k=1}^{N} (r_k^2 - \|x - p_k\|^2)^2. \]

Note:
What does this imply for the measurement noise? Several *ad hoc* solutions exist for transforming the nonlinear problem into a linear problem.
DOA Triangulation

Angle observations from sensor at position $p_k$

\[ \varphi_k = \arctan\left(\frac{x_2 - p_{k,2}}{x_1 - p_{k,1}}\right) \]

\[(x_1 - p_{k,1}) \tan(\varphi_k) = x_2 - p_{k,2}\]

Linear model immediate

\[ y = hx + e \]

\[ y = \begin{pmatrix} p_{1,1} \tan(\varphi_1) - p_{1,2} \\ p_{2,1} \tan(\varphi_2) - p_{2,2} \\ \vdots \\ p_{N,1} \tan(\varphi_N) - p_{N,2} \end{pmatrix} \]

\[ h = \begin{pmatrix} \tan(\varphi_1) & -1 \\ \tan(\varphi_2) & -1 \\ \vdots \\ \tan(\varphi_N) & -1 \end{pmatrix} \]
Received signal strength (RSS) observations:

- All waves (radio, radar, IR, seismic, acoustic, magnetic) decay exponentially in range
- Receiver $k$ measures energy/power/signal strength for wave $i$:
  \[ P_{k,i} = P_{0,i} \| x - p_k \|^{\beta_i} \]
- Transmitted signal strength and path loss constant unknown
- Communication constraints make coherent detection from the signal waveform impossible
- Compare $P_{k,i}$ for different receivers
Log model:

\[
\tilde{P}_{k,i} = \bar{P}_{0,i} + \beta_i \log(||x - p_k||) =: c_k(x)
\]

\[
y_{k,i} = \tilde{P}_{k,i} + e_{k,i}
\]

Use separable least squares to eliminate path loss constant and transmitted power for wave \( i \):

\[
\hat{(x, \theta)} = \arg \min_{x, \theta} V(x, \theta)
\]

\[
V(x, \theta) = \sum_{i=1}^{M} \sum_{k=1}^{N} \left( \frac{y_{k,i} - h(c_k(x), \theta_i))}{\sigma_{P,i}^2} \right)^2
\]

\[
h(c_k(x), \theta_i) = \theta_{i,1} + \theta_{i,2} c_k(x)
\]

\[
c_k(x) = \log(||x - p_k||)
\]

Finally, use NLS to optimize over 2D target position \( x \).
Network with 10 microphones around a ‘camp’. Shooter aiming inside the camp. Supersonic bullet injects a shock wave followed by an acoustic muzzle blast. Fusion of time differences give shooter position and aiming point.

Model, with $x$ being the position of the shooter,

$$y_{k}^{\text{MB}} = t_0 + b_k + \frac{1}{c} \| p_k - x \| + e_{k}^{\text{MB}},$$

$$y_{k}^{\text{SW}} = t_0 + b_k + \frac{1}{r} \log \frac{v_0}{v_0 - r \| d_k - x \|} + \frac{1}{c} \| d_k - p_k \| + e_{k}^{\text{SW}},$$

where $d_k$ is an implicit function of the state.
Shooter Localization: results

- WLS loss function illustrates the information in $y_k^{MB} - y_k^{SW}$, $k = 1, 2, \ldots, N = 10$
- Both shooter position and aiming direction $\alpha$ are well estimated for each shot.
- Also bullet’s muzzle speed and ammunition length can be estimated!
Signal model for localization in sensor networks

\[ y = h(x - p) + e, \quad \text{cov}(e) = R \]

\( x \) is the unknown position, \( p \) is the (known in this chapter) sensor locations.

The basic network measurements:

- **TOA** \( r_k = \|x - p_k\| + e_k \)
- **TDOA** \( r_k = \|x - p_k\| + r_0 + e_k \)
- **DOA** \( \varphi_k = \arctan2(x_2 - p_{k,2}, x_1 - p_{k,1}) + e_k \)
- **RSS** \( y_k = P^0 - \beta \log(\|x - p_k\|) \)

NLS or NLT general approaches to estimate \( x \)
Tricks (not statistically optimal!)

**TOA** range parameter trilateration: $r_k^2$ is linear in $(x_1, x_2, x_1^2 + x_2^2)^T$

**TOA** reference sensor trilateration: $r_k^2 - r_1^2$ is linear in $x$

**DOA** triangulation approach: $x$ is an affine function in $\tan(\varphi_k)$