Solution for TSRT09 Control Theory, 2022-08-23

1. (a) An unstable pole gives a lower bound on the bandwidth of the closed loop system, while instead an unstable zero gives an upper bound on the bandwidth. When the zero is to the left of the pole then the two bounds are not compatible and it is difficult to obtain good performances (and sometimes even stability) for the closed loop system.

   (b) A white noise has a constant spectrum: $\Phi_u(\omega) = \Phi_0$. Hence for the transfer function $G(s) = \frac{3}{s+5}$, it is $\Phi_y = |G(i\omega)|^2\Phi_u(\omega) = \frac{9\Phi_o}{\omega^2+25}$.

   (c) The sensitivity function can be written as $S = \frac{1}{1+GF_y}$. For our case it must be $S \to 1 (= 0\,dB)$ when $\omega \to \infty$. Only the Bode diagram (b) satisfies this property. Furthermore, the system satisfies the conditions for the Bode integral theorem, hence the Bode diagram has to have equal area above and below 0 (the system is stable). Only the Bode diagram (b) satisfies to this property.

   (d) It is enough to choose $u = -x_1 - x_2^3 + r$. In fact this gives

$$\dot{y} = \dot{x}_1 = -x_1 + r = -y + r$$

   The associated zero dynamics is

$$\dot{x}_2 = -x_2 - x_1 - x_2^3 + r$$

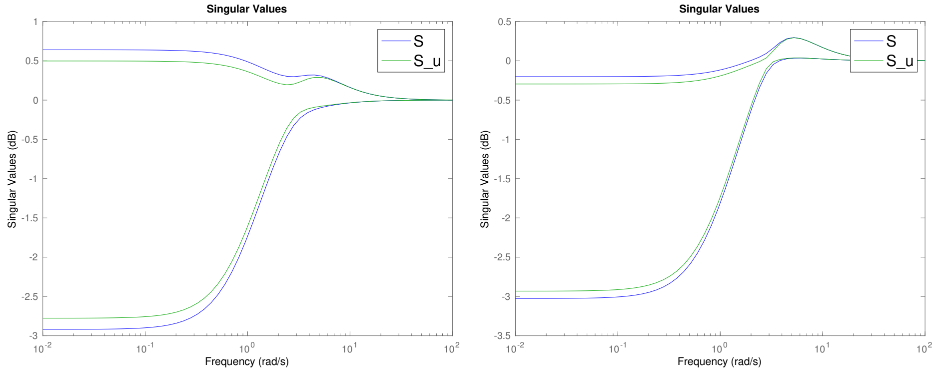
2. (a) It can be checked that both regulators $F_{y,1}$ and $F_{y,2}$ give a stable closed loop, as well as internal stability. Computing RGA of G at any frequency ω it is

$$\text{RGA}(G(i\omega)) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

   which suggests that the coupling $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$ should be used. Hence the regulator $F_{y,2}$ is the one achieving the best performances from a decoupling perspective. In addition, one can compute the sensitivity function of the closed loop for the two $F_{y,i}$, see Fig. 1, and observe that both S and S_u have a better low frequency behavior for $F_{y,2}$: the largest singular value is lower in both S and S_u. Putting everything together $F_{y,2}$ is the best controller in this case.

   (b) See Fig. 1.

   (c) The singular values of S are given by the square roots of the eigenvalues of S. At $w = 0.1\,\text{rad/sec}$, for $F_{y,1}$ their product is 0.7708 and idem for the product of the singular values of S_u, while for $F_{y,2}$ the product is 0.6913 for both S and S_u.



(a) (b)

Figure 1: Singular values of S and S_u for the two regulators $F_{y,1}$ (left) and $F_{y,2}$ (right).

(d) From $SG = GS_u$, when G, S and S_u are square and for each frequency ω at which $G(i\omega)$ is invertible, it is $S = GS_uG^{-1}$, i.e., S and S_u are related by a similarity transformation, which preserves the eigenvalues. Therefore also the singular values of S and S_u are the same, and so is their product. This is also rather evident from Fig. 1.

3. (a) Consider the closed loop system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - Kx_1 - K^3x_1^3$$

Equilibrium points must obey to $\dot{x}_1 = \dot{x}_2 = 0$. This leads to $x_2 = 0$ and $Kx_1 \underbrace{(1 + K^2x_1^2)}_{>0} = 0$, which implies $x_1 = 0$. Hence the origin $x = 0$ is the only equilibrium point. The linearization at 0 of the closed loop system gives the matrix

$$\begin{bmatrix} 0 & 1 \\ -K & -1 \end{bmatrix}$$

of eigenvalues

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4K}}{2}$$

Concerning stability and character of the equilibrium, we have the following cases:

- for $K < 0$ (i.e., positive feedback), the equilibrium $x = 0$ is saddle point, hence unstable;
- for $0 < K < 1/4$, the equilibrium $x = 0$ is a stable node;
- for $K = 1/4$, the equilibrium is a stable node with two identical eigenvalues (equal to -1/2). The associated Jordan form is

$$J = \begin{bmatrix} -1/2 & 1 \\ 0 & -1/2 \end{bmatrix}$$

meaning that there is only 1 eigenvector;

- for $K > 1/4$, the equilibrium $x = 0$ is a stable focus.

(b) Differentiating V along the trajectories of the system, one gets

$$\dot{V} = 2ax_1x_2 + 4bx_1^3x_2 + 2cx_2(-Kx_1 - K^3x_1^3 - x_2)$$

If one chooses

$$c > 0, \quad a = cK, \quad b = cK^3/2$$

then $\dot{V} = -2cx_2^2 \leq 0$. Hence the system is at least stable. To show asymptotic stability, notice that the trajectories of the closed loop system staying on the level surfaces of the Lyapunov function (i.e., those for which $\dot{V} = 0$) have to obey $x_2 = 0 \; \forall t$, which implies $x_1 = 0 \; \forall t$ (since, as already mentioned, $1 + K^2x_1^2 > 0$ always). Hence the only trajectory living in the level surface is $x = 0$, which proves asymptotic stability. Since the origin is the only equilibrium point, stability is global.

4. The constraints can be satisfied by giving different weights to the states and inputs in a LQ design. For instance, choosing

$$Q_1 = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_2 = 1$$

the LQ design `L=lqr(A,B,Q1,Q2)` gives

$$L = \begin{bmatrix} 7.0671 & 2.7569 & 0.4649 \end{bmatrix}$$

The evolution of the closed loop system $\dot{x} = (A - BL)x$ from $x(0)$ is shown in Fig.2. All constraints are satisfied.

5. (a) The describing function $Y_f(C)$ for the ideal relay is given in Example 14.1 of the book, and it is always real. Since $r = 0$, the linear part of the system can be written as

$$\tilde{G}(S) = G(S)F(S) = \frac{K(\tau s + 1)}{\tau s(s+2)(s+3)}$$

whose Nyquist curve is given in Fig. 3 for $K = 1$ and $K = 9.42$ (red curves). Regardless of the value of K, there is always an intersection between $-1/Y_f(C) = -\frac{\pi C}{4}$ and $\tilde{G}(i\omega)$, see Fig. 3.

(b) The frequency of the oscillations can be obtained by computing the value of ω for which $\tilde{G}(i\omega)$ is real (since $-1/Y_f(C)$ is real). Doing the calculations:

$$\tilde{G}(i\omega) = \frac{10K(i\omega/10 + 1)}{i\omega(i\omega+2)(i\omega+3)} = \frac{10K\left(-5\omega + 0.1\omega(6-\omega^2) - i(6 - 0.5\omega^2)\right)}{\omega(25\omega^2 + (6-\omega^2)^2)}$$

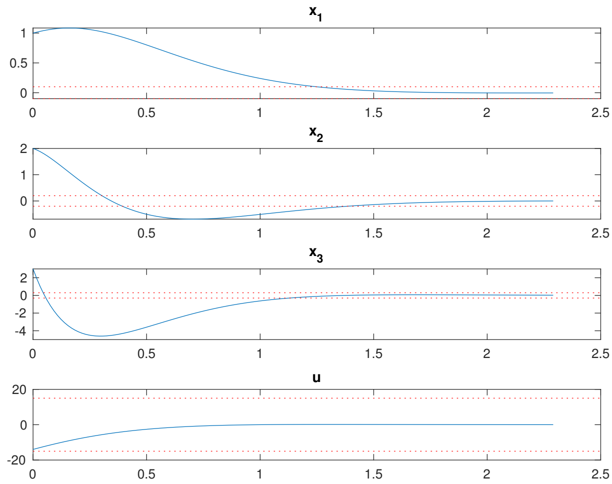


Figure 2: Simulation of the system of Ex. 4. From top to bottom: x_1, x_2, x_3 and u. The constraints are shown in red.

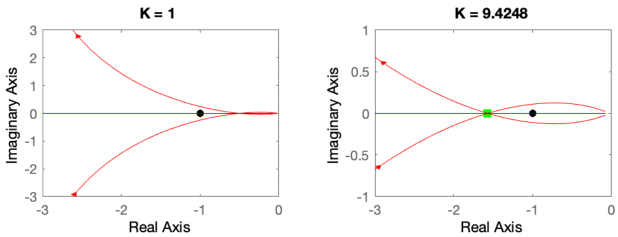


Figure 3: Ex. 5: Describing function $-\frac{1}{Y_f(C)}$ (blue) and Nyquist curve of the loop gain $\tilde{G}(i\omega)$ (red) for two values of K. The green dot in the right panel corresponds to $-1/Y_f(2)$.

This expression is real when the imaginary term vanishes, i.e., for $\bar{\omega} = 2\sqrt{3}$. The frequency does not change with K.

(c) Since $-1/Y_f(C)$ grows by moving from right to left, as in Fig. 14.9(a) of the book, the oscillations are stable.

(d) An amplitude of $C = 2$ of the oscillation corresponds to $-\frac{1}{Y_f(2)} = -1.5708$. At $\bar{\omega} = 2\sqrt{3}$, compute the real part of $\tilde{G}$: $\tilde{G}(i\bar{\omega}) = -\frac{5.6K}{33.6}$. Hence it must be $K \leq 9.4248$, see right panel in Fig. 3.