

## Solution for TSRT09 Control Theory, 2022-06-08

1. (a) A system is controllable if all its poles can be allocated arbitrarily by a state feedback; it is stabilizable if all its unstable poles (but not necessarily the stable ones) can be allocated arbitrarily by state feedback. The PBH test can be used for checking both properties.
- (b) The case  $p > z$  is the most difficult, as according to eq. (7.31) and (7.35) in the course book, the cross-over frequency (and hence the bandwidth of the closed loop system) must be less than  $z$  and larger than  $p$ . When  $p > z$ , this results in poor behavior for the sensitivity function and for the complementary sensitivity function.
- (c) The transfer function (iii) has poles in  $-0.2 \pm i3.995$  meaning that it has a resonance peak around 4 rad/sec.
- (d) Let  $G(s) = \frac{1}{s+2}$  and  $G_d(s) = \frac{2}{s+3}$ . For perfect disturbance cancellation it must be  $u = -G^{-1}G_d v$ . If the disturbance is sinusoidal,  $v(t) = 2 \sin(3t)$ , also  $u$  becomes sinusoidal, with a gain factor of  $|G^{-1}(i3)G_d(i3)|$ . We then get the constraint  $u_0 \geq |G^{-1}(i3)G_d(i3)|2 = \frac{2}{3}\sqrt{26} \approx 3.40$ .

2. (a) The transfer function of the system is

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{2s+3}{(s+2)(s+1)} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2s+5}{(s+2)(s+3)} \end{bmatrix}$$

meaning that the pole polynomial is  $p(s) = (s+1)(s+2)(s+3)$ . The poles are  $-1$ ,  $-2$ ,  $-3$ , and the zero is  $-2$ .

- (b)  $\|G\|_\infty = 1.7676$  (or 4.94 dB).
- (c) The RGA at 0 frequency is

$$\text{RGA}(G(0)) = G(0) \odot (G^{-1}(0))^T = \begin{bmatrix} 1.2500 & -0.2500 \\ -0.2500 & 1.2500 \end{bmatrix}$$

- (d) The sensitivity function is

$$\begin{aligned} S(i\omega) &= (I + GF)^{-1} \\ &= \frac{1}{s^3 + 86s^2 + 1531s + 2686} \begin{bmatrix} s^3 + 46s^2 + 151s + 106 & -20s^2 - 80s - 60 \\ -20s^2 - 80s - 60 & s^3 + 46s^2 + 191s + 186 \end{bmatrix} \end{aligned}$$

- (e)  $S$  is the transfer function from disturbance  $w$  to control error (here equal to  $-y$ ), hence looking at the singular values of  $S$ , shown in Figure 1, one can see that the gain from  $w$  to control error is  $< -20\text{dB}$  for  $\omega < 1$  rad/sec.

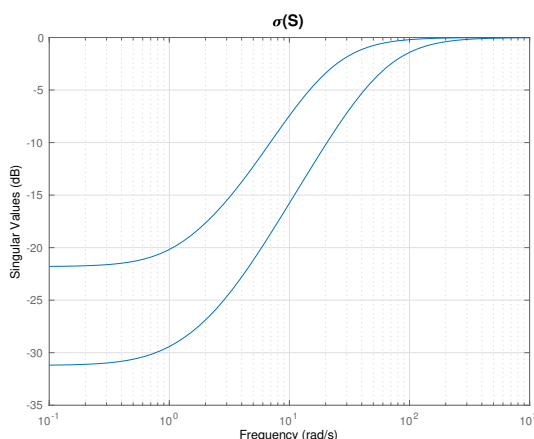


Figure 1: Singular values for  $S$  in Exercise 2.

- (f) The disturbance is constant, hence we have to look at  $\omega = 0$ . Computing the “best” ratio  $w_1/w_2$  corresponds to computing the eigenvector of the largest singular value of  $S(0)$  (i.e., eigenvalue of  $\sqrt{S^*(0)S(0)}$ ). Since  $\sigma(S(0)) = \{0.0275, 0.0812\}$ , we have to look for the eigenvector associated to 0.0812, which is  $w = [-0.4719 \ 0.8817]^T$ , meaning that the sought ratio is  $w_1/w_2 = -0.5352$ .

3. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0], \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} 48 & 0 \\ 0 & 1 \end{bmatrix}$$

- (a) When the observer gain is  $K_1 = [2 \ 1]^T$ , denoting  $\hat{x}_1$  the state observer, then

$$\dot{\hat{x}}_1 = (A - K_1 C)\hat{x}_1 + K_1 y$$

and the eigenvalues of  $A - K_1 C$  are both equal to -2.

- (b) If  $\tilde{x}_1 = x - \hat{x}_1$ , the estimation error ODE is

$$\dot{\tilde{x}}_1 = (A - K_1 C)\tilde{x}_1 + \begin{bmatrix} N & -K_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The associated covariance matrix  $P_1 = E\tilde{x}_1\tilde{x}_1^T$  can be obtained by solving a Lyapunov equation (in matlab: `P1=lyap(A-K1*C, [N -K1]*R*[N'; -K1']`):

$$P_1 = \begin{bmatrix} 2.2812 & 4.8438 \\ 4.8438 & 19.6562 \end{bmatrix}$$

- (c) The Kalman filter (here denoted  $K_2$ , of error covariance  $P_2$ ) can be computed in matlab using `[K2 P2,pole2]=lqe(A,N,C,R1,R2,0)`, which gives

$$K_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix} \quad \text{and poles} = -2 \pm 1.732i$$

- (d) The variance of the estimation error,  $\Sigma_i = E[\tilde{x}_i^T \tilde{x}_i]$ , is nothing but the trace of  $P_i$ , hence

$$\Sigma_1 = \text{trace}(P_1) = 21.9375 > \Sigma_2 = \text{trace}(P_2) = 18$$

as expected.

4. (a) From Chapter 14 in the book, the describing function for the relay with hysteresis is

$$Y_f(C) = \frac{4}{\pi C} \left( \sqrt{1 - \frac{1}{4C^2}} - \frac{i}{2C} \right), \quad C \geq 0.5$$

$$-1/Y_f(C) = -\frac{\pi C}{4} \sqrt{1 - \frac{1}{4C^2}} - i\frac{\pi}{8}.$$

The frequency function for the system is

$$G(i\omega) = \frac{10}{i\omega(1+i\omega)} = \frac{-10(\omega+i)}{\omega(\omega^2+1)}.$$

The intersection is obtained by solving the system of equations (in  $\omega$  and  $C$ ):  
 $G(i\omega) = -1/Y_f(C)$ ;

$$\text{Imaginary part: } \frac{10}{\omega(\omega^2+1)} = \frac{\pi}{8}$$

$$\text{Real part: } \frac{10}{\omega^2+1} = \frac{\pi}{8} \sqrt{4C^2-1}.$$

The solution is  $\omega_o = 2.8288$  and  $C_o = 1.5002$ . This can be checked also graphically, see Fig. 2.

- (b) The curve  $-1/Y_f(C)$  grows moving from right to left.

$$C < C_o \implies -1/Y_f(C) \text{ is encircled by } G(i\omega) \implies \text{oscillation grows}$$

$$C > C_o \implies -1/Y_f(C) \text{ is not encircled by } G(i\omega) \implies \text{oscillation decreases}$$

Summing up: the oscillation is stable in amplitude.

5. (a) Choosing state space variables  $x_1 = \int e$  and  $x_2 = y$ , together with  $u = \text{sat}(x_1 + e)$  and  $e = -y$  (since  $r = 0$ ), one gets the nonlinear state space model

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = \text{sat}(x_1 - x_2)$$

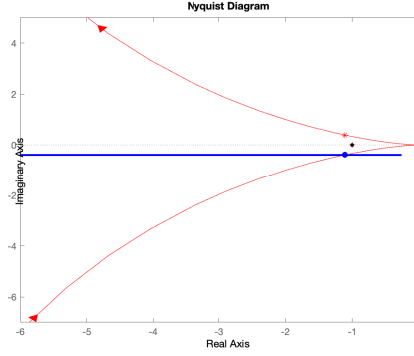


Figure 2: Exercise 4. In red:  $G(i\omega)$ , in blue  $-\frac{1}{Y_f(C)}$ . Solid blue point: intersection.

which corresponds to a system switching between 3 different right hand sides in 3 different regions of the state space:

**A:** for  $|x_1 - x_2| \leq 1$ , it is

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

**B:** for  $x_1 - x_2 < -1$  (i.e.,  $x_2 > x_1 + 1$ ) it is

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= -1 \end{aligned}$$

**C:** for  $x_1 - x_2 > 1$  (i.e.,  $x_2 < x_1 - 1$ ) it is

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= 1 \end{aligned}$$

Only the first region admits an equilibrium point corresponding to  $x = 0$ . Since the eigenvalues of  $A$  are  $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ , the equilibrium point is a stable focus.

Phase portrait:

- A:** The trajectories spiral down to the origin;
- B:** It is  $\frac{\dot{x}_1}{\dot{x}_2} = \frac{dx_1}{dx_2} = x_2$ , whose solution  $x_1 = \frac{1}{2}x_2^2 + c$  is a parabola. Furthermore,  $x_2$  keeps decreasing, while  $x_1$  increases for  $x_2 < 0$  and decreases otherwise;
- C:** The solution in this case is  $x_1 = -\frac{1}{2}x_2^2 + c$ , and  $x_2$  keeps growing, while  $x_1$  decreases if  $x_2 > 0$  and increases otherwise.

The overall phase portrait can be seen in Fig. 3(a).

- (b) Similarly to part (a), there are 3 regions divided by the lines  $x_1 - x_2 = -1$  and  $x_1 - x_2 = 1$ , and characterized by the 3 systems

**A:** for  $|x_1 - x_2| \leq 1$ , same as before

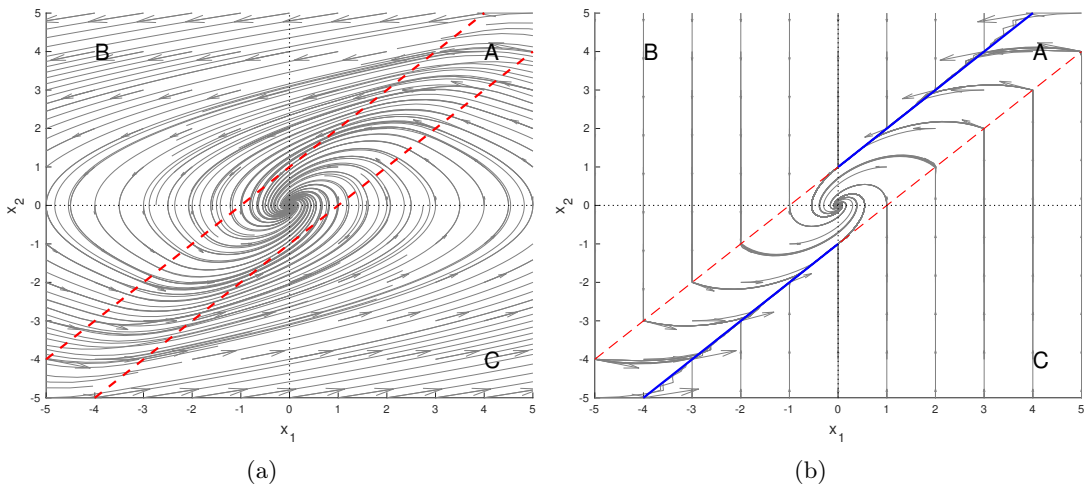


Figure 3: Phase portraits for Exercise 5.

**B:** for  $x_1 - x_2 < -1$

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 &= -1\end{aligned}$$

**C:** for  $x_1 - x_2 > 1$

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 &= 1\end{aligned}$$

The new phase portrait is in Figure 3(b).

To show chattering behavior, let us look at what happens as the trajectories cross the dividing lines. Let us consider first  $x_1 - x_2 = -1$ , for  $x_1 > 0$  (and  $x_2 > 1$ ). A trajectory coming from the region **A** has  $\dot{x}_1 = -x_2 < -1$  and  $\dot{x}_2 = x_1 - x_2 = -1$ , which implies that it is pushed out of the region **A** towards **B**. But then in **B** the system becomes  $\dot{x}_1 = 0$  and  $\dot{x}_2 = -1$  i.e., the trajectory is forced back into **A**. This is a chattering behavior: the value of  $\dot{x}_1$  switches very quickly between 0 and  $-x_2 < -1$ . Something similar does not happen when  $x_1 < 0$  (and  $x_2 < 1$ ) where instead the trajectories enter from **B** to **A** and continue in **A** afterwards, see Figure 3(b).

A specular behavior occurs in the other dividing line  $x_1 - x_2 = 1$ .

Overall, chattering occurs on the blue lines shown in Figure 3(b). What happens in practice in a simulator is that the trajectory slides along the blue line towards the origin until it reaches the vertical axis. After that it enters the region **A** definitively and spirals towards the origin.