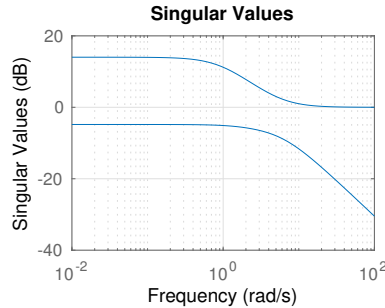


Solution for TSRT09 Control Theory, 2022-03-25

1. (a) See Chapter 12 of the book.
- (b) For $G(s)$:
 - i) The pole polynomial is $p(s) = (s + 1)(s + 3)^2(s + 5)$.
 - ii) The 2 singular values are given in the following plot:



- iii) From

$$\text{RGA}(G(0)) = \begin{bmatrix} 1.0385 & -0.0385 \\ -0.0385 & 1.0385 \end{bmatrix}$$

the pairing is $y_1 \leftrightarrow u_1$ and $y_2 \leftrightarrow u_2$.

- iv) Since the order of the pole polynomial is 4, that is also the dimension of a minimal state space realization:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1.5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Since (A, B) is controllable, it is also stabilizable. Since (A, C) is observable, it is also detectable.

2. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & 0.1 \\ 0 & \frac{10(s+2)(s+1)}{(s+0.1)(s+20)^2} \\ \frac{1}{s+1} & \frac{1}{(s+0.1)(s+20)} \\ 1 & \frac{s+1}{(s+0.1)(s+20)} \end{bmatrix}$$

Using the function `hinfsyn` in matlab, an \mathcal{H}_∞ controller can be designed, with the function call

```
[Fy_pos, CL, GAMMA, INFO] = hinfsyn(Ge, 1, 1)
Fy = - Fy_pos
```

The optimal \mathcal{H}_∞ regulator corresponds to the transfer function

$$F_y = \frac{34.19s^3 + 1371s^2 + 1.381 \cdot 10^4s + 1368}{s^4 + 113.9s^3 + 1038s^2 + 2406s + 1481}$$

and $\gamma = 0.7568$. $S(i\omega)$, $T(i\omega)$ and $W_u(i\omega)$ are shown in Fig. 1, together with the corresponding bounds $\gamma W_*^{-1}(\omega)$. Only $S(i\omega)$ is critical in this \mathcal{H}_∞ design.

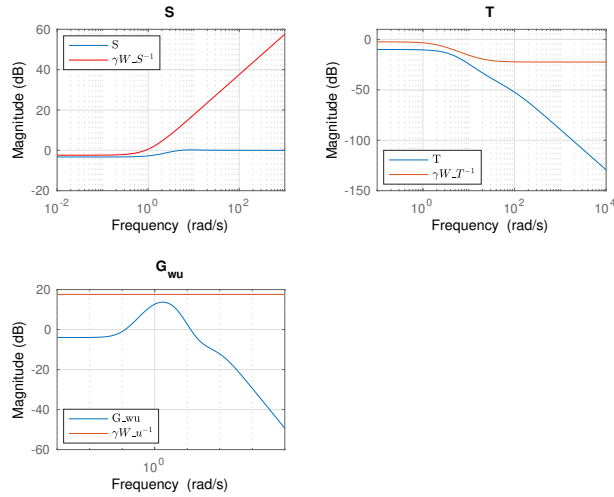


Figure 1: Exercise 2: \mathcal{H}_∞ design. In blue are S , T , and G_{wu} . In red are $\gamma W_S^{-1}(\omega)$, $\gamma W_T^{-1}(\omega)$, $\gamma W_u^{-1}(\omega)$.

- (b) The sensitivity S at low frequencies can be decreased in many ways, for instance increasing the gain in W_S or moving the pole of W_S to lower frequency. Fig. 2 shows the \mathcal{H}_∞ design for the following choices of W_S :

Fig. 2 left panel: $W_S = \frac{10}{s+1}$

Fig. 2 right panel: $W_S = \frac{1}{s+0.1}$

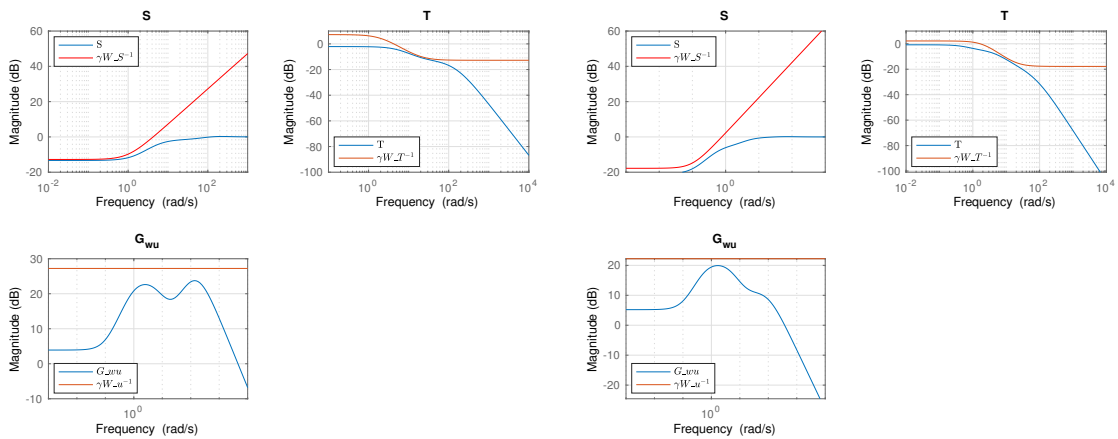


Figure 2: Exercise 2: \mathcal{H}_∞ design with improved S .

3. (a) When the input is a constant $u = u_0 > 0$ the system is

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 \\ \dot{x}_2 &= -x_2 + u_0\end{aligned}$$

To find the equilibria in $x_1 > 0$, $x_2 > 0$, put $\dot{x} = 0$ and solve the equations. This gives that

$$\bar{x} = \begin{bmatrix} \sqrt{u_0} \\ u_0 \end{bmatrix}, \quad \bar{u} = u_0$$

is the only admissible equilibrium point (in $\mathbb{R}_{\geq 0}^2$). The linearization at (\bar{x}, \bar{u}) is

$$A = \frac{\partial f}{\partial x}(\bar{x}, u_0) = \begin{bmatrix} -2x_1 & 1 \\ 0 & -1 \end{bmatrix} \Big|_{(\bar{x}, \bar{u})} = \begin{bmatrix} -2\sqrt{u_0} & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \frac{\partial f}{\partial u}(\bar{x}, u_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The associated linear system is

$$\dot{z} = Az + Bu \quad \text{where} \quad z = x - \bar{x}$$

This system has eigenvalues $\lambda_1 = -2\sqrt{u_0}$ and $\lambda_2 = -1$, with associated eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2\sqrt{u_0} - 1 \end{bmatrix}$$

meaning that the equilibrium (\bar{x}, u_0) is a stable node with two distinct eigenvectors when $u_0 \neq 1/4$. When instead $u_0 = 1/4$, then $\lambda_1 = \lambda_2 = -1$, and the Jacobian linearization is

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

which is already in Jordan form: it has one block of dimension 2, meaning that there is only 1 eigenvector in this case, equal to v_1 . The original nonlinear system also has a stable node at (\bar{x}, u_0) , with local phase portrait that depends on the value of u_0 (which “rotates” v_2).

- (b) Choosing $u_0 = 1$ then one can sketch a phase portrait for the original system near the equilibrium point by looking at the direction of the two eigenvectors, which now are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These are drawn in red in Fig. 3. Since $\lambda_1 = -2 < \lambda_2 = -1$, v_1 is the “fast” direction and v_2 is the “slow” one (trajectories are aligned with v_2 near \bar{x}). Some other information can be added by looking at dx_2/dx_1 :

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 + 1}{-x_1^2 + x_2} \Rightarrow \frac{dx_2}{dx_1} \rightarrow \begin{cases} 0 & \text{if } x_2 = 1 \text{ (horizontal line)} \\ -1 & \text{if } x_1 \text{ is fixed and } x_2 \rightarrow \infty \end{cases}$$

and some more can be obtained by simulating the system from different initial conditions, see Fig. 3.

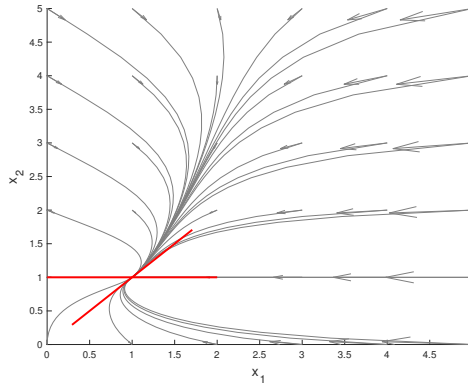


Figure 3: Exercise 3: phase portrait for $u_0 = 1$. The red lines are the eigenvectors directions around \bar{x} .

4. (a) From Chapter 14 in the book, the describing function for the relay with deadzone is the real function

$$Y_f(C) = \frac{4}{\pi C} \sqrt{1 - \frac{1}{C^2}}, \quad C > 1$$

i.e.,

$$-1/Y_f(C) = -\frac{\pi C}{4} \sqrt{\frac{C^2}{C^2 - 1}} \quad C > 1$$

and is shown in Fig. 4 (top left panel).

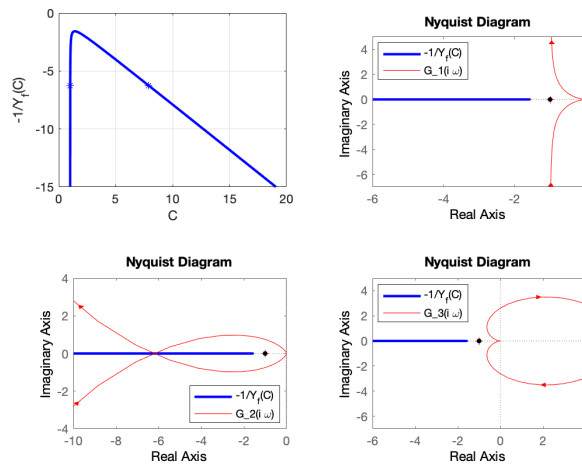


Figure 4: Exercise 4. Top left panel: $-1/Y_f(C)$ versus C . The two stars are the intersection points for $G_2(i\omega)$. Remaining panels: $-1/Y_f(C)$ and $G_i(i\omega)$, $i = 1, 2, 3$. In red: $G(i\omega)$, in blue $-\frac{1}{Y_f(C)}$.

Drawing the Nyquist plots of the 3 TF (see remaining panels in Fig. 4), one can see that only $G_2(i\omega)$ intersects $-1/Y_f(C)$, hence only $G_2(i\omega)$ can admit self-sustained oscillations.

- (b) From the shape of $-1/Y_f(C)$ in the top left panel of Fig. 4, there must be 2 intersections. In fact, the curve $-1/Y_f(C)$ in the complex plane first moves right along the real axis and then moves left when C grows, and its starting point when C is very close to 1 is arbitrarily to the left ($C = 1$ is a singularity, in which $-1/Y_f(C) = -\infty$). (You might [and should!] remember this from Lab 3 where something similar was computed...). From Fig. 14.9 in the book, one oscillation is stable in amplitude, the other unstable.

Even though it is not requested by the exercise, the intersection points are

$$\omega = 2, \quad C_1 = 1.0081, \quad C_2 = 7.8936$$

and are shown as stars in the top left panel of Fig. 4. They correspond to $G_2(i2) = -1/Y_f(C_j) = 6.25$.

5. (a) Differentiating the output: $\dot{y} = \dot{x}_1 = x_2$, $\ddot{y} = \dot{x}_2 = \sin x_3 + u$, i.e., the relative degree is $\nu = 2$.
- (b) Choosing $u = -\sin x_3 - ax_1 - bx_2 + r$ we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= r - ax_1 - bx_2 \\ \dot{x}_3 &= x_1^3 + x_2 + x_3 \\ y &= x_1 \end{aligned}$$

which gives a linear relationship between r and y (involving only x_1 and x_2 , not x_3), with transfer function $G(s) = \frac{1}{s^2 + bs + a}$, and poles that can be placed arbitrarily. Choosing $a = b = 1$ we obtain the desired transfer function. Hence the sought regulator is $u = -\sin x_3 - x_1 - x_2 + r$.

- (c) The system with this regulator has a zero dynamics given by $\dot{x}_3 = x_3$, which is unstable.
- (d) Let us compute for which values of α, β, γ we get a relative degree equal to $\nu = 3$.

Differentiating the output $\bar{y} = h(x)$

$$\begin{aligned}
\dot{\bar{y}} &= \alpha \dot{x}_1 + \beta \dot{x}_2 + \gamma \dot{x}_3 \\
&= \alpha x_2 + \beta \sin x_3 + \beta u + \gamma x_1^3 + \gamma x_2 + \gamma x_3 \quad / \text{ assume: } \beta = 0 / \\
&= \gamma x_1^3 + (\alpha + \gamma)x_2 + \gamma x_3 \\
\ddot{\bar{y}} &= \gamma 3x_1^2 \dot{x}_1 + (\alpha + \gamma)\dot{x}_2 + \gamma \dot{x}_3 \\
&= \gamma 3x_1^2 x_2 + (\alpha + \gamma)(\sin x_3 + u) + \gamma(x_1^3 + x_2 + x_3) \quad / \text{ assume: } \alpha = -\gamma / \\
&= \gamma 3x_1^2 x_2 + \gamma(x_1^3 + x_2 + x_3) \\
\dddot{\bar{y}} &= \gamma(6x_1 \dot{x}_1 x_2 + 3x_1^2 \dot{x}_2 + 3x_1^2 \dot{x}_1 + \dot{x}_2 + \dot{x}_3) \\
&= \underbrace{\gamma(6x_1 x_2^2 + (3x_1^2 + 1) \sin x_3 + 3x_1^2 x_2 + x_1^3 + x_2 + x_3)}_{L_f^3 h(x)} + \underbrace{\gamma(3x_1^2 + 1) u}_{L_g L_f^2 h(x)}
\end{aligned}$$

where we have used the notation of Lie derivative from Ch. 17 of the book. Hence the relative degree is 3 for $\alpha = -\gamma$, $\gamma \neq 0$, and $\beta = 0$. The change of input

$$u = \frac{-L_f^3 h(x) + v}{L_g L_f^2 h(x)}$$

linearizes the relationship between v and \bar{y} (i.e., $\dddot{\bar{y}} = v$), and the feedback

$$v = r - x_1 - x_2 - x_3$$

leads to the sought transfer function between r and \bar{y} .