Solution for TSRT09 Control Theory, 2021-08-24

1. (a) $\text{RGA}(G(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that is, we can pair u_1 with y_1 and u_2 with y_2.

   (b) The system $\dot{x} = -3x + v$ is stable, hence we can use Theorem 5.3 of the book: $A\Pi_x + \Pi_x A^T + BRB^T = 0$, where Π_x is the variance of x. With $A = -3, B = 1, R = 1$ it is $\Pi_x = 1/6$.

   (c) The system is controllable if the controllability matrix $\mathcal{S} = \begin{bmatrix} B & AB \end{bmatrix}$ has full rank (here rank $= 2$). Since B is already full rank, the system is controllable for all values of a, b, c, d.

   (d) The two closed loop systems corresponding to L_1 and L_2 are

$$A - BL_1 = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \quad \text{and} \quad A - BL_2 = \begin{bmatrix} -5 & -2 \\ 0 & -5 \end{bmatrix}$$

   which have the same eigenvalues. In the first closed loop system decoupling is achieved: $A - BL_1$ diagonal implies that $G(s) = I(sI - (A - BL_1))^{-1}I$ is also diagonal.

2. (a) Using the state variables mentioned in the text:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -0.1x_2 - 3x_1 + 4x_1^3$$

   Linearization around the origin gives $\dot{z} = Az = \begin{bmatrix} 0 & 1 \\ -3 & -0.1 \end{bmatrix} z$ where A has eigenvalues $\lambda = -0.05 \pm 1.7313i$, i.e., 0 is a stable focus.

   (b) The minors of the system are $\frac{2s+1}{(s+1)(s+3)}, \frac{1}{s+1}, \frac{1}{s+3}, \frac{1}{s+4}, \frac{s-3}{(s+1)(s+3)(s+4)}$. The pole polynomial is therefore $p(s) = (s+1)(s+3)(s+4)$. The maximal minor is $\frac{s-3}{p(s)}$. Poles are $-3, -2, -1$ and there is a zero in $+3$. The zero is non-minimum phase, hence in the closed loop system there is an upper bound on the bandwidth equal approx. to $3/2 = 1.5$ rad/s.

   (c) The spectral factorization of $\phi(\omega)$ gives the linear, stable system $G_w(s) = \frac{1}{s+3}$ meaning that $w = G_w(s)v$ where v is a white noise of intensity 1. Expanding the state with $x_2 = w$, so that $\dot{x}_2 = -3x_2 + v$, results in $\dot{x} = Ax + Bu + Nv$ where

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. (a) The constraints can be satisfied by giving different weights to the states and inputs in a LQ design. For instance, choosing

$$Q_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

the LQ design `L=lqr(A,B,Q1,Q2)` gives

$$L = \begin{bmatrix} 0.3586 & 0.2657 & 0.2298 \\ 2.5777 & 1.9858 & 1.4845 \end{bmatrix}$$

The evolution of the closed loop system $\dot{x} = (A - BL)x$ from $x(0)$ is shown in Fig.1. All constraints are satisfied.

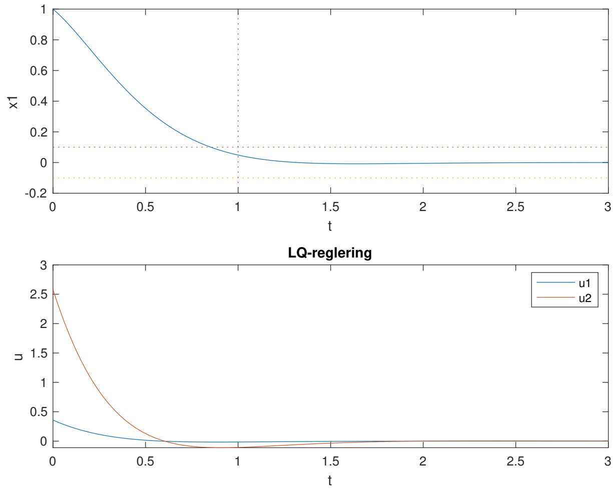

Figure 1: Simulation of the system of Ex. 3. Upper plot: x_1; lower plot u_1 and u_2.

(b) Adding a Kalman filter can improve the result if there is measurement error in the sensors that measure the states, and can compensate for a possible sensor failure. On the other hand, unlike an LQ regulator, an LQG regulator is not robust.

4. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{s+4}{(s+1)(s+3)^2} \\ \frac{1}{s+2} & \frac{1}{(s+1)(s+2)(s+3)} \\ 1 & \frac{1}{(s+1)(s+3)} \end{bmatrix}$$

which corresponds to the state space model (after elimination of redundant zero/poles with `minreal`)

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \\ y &= Cx + w \end{aligned}$$

with

$$A = \begin{bmatrix} -2.2282 & -0.8597 & -1.6732 & -3.2377 \\ 0.3073 & -2.4361 & 0.7207 & 0.2242 \\ -0.1021 & 0.4690 & -2.5893 & 0.0153 \\ -0.2872 & -0.0285 & 0.3115 & -1.7463 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5997 \\ -0.0242 \\ 0.3698 \\ 0.5030 \end{bmatrix}, \quad N = \begin{bmatrix} -0.0657 \\ -0.4961 \\ -0.3749 \\ 0.3301 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.6334 & -0.4688 & 0.8357 & 0.1184 \\ -0.1314 & -0.9923 & -0.7498 & 0.6602 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.3880 & 0.1752 & 0.1523 & 0.3591 \end{bmatrix}$$

The condition $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is satisfied, and $A - NC$ is stable, hence the model is in innovation form (i.e., N is the Kalman gain).

(b) An $\mathcal{H}_\infty$ controller can be designed using the function `hinfsyn` in matlab. In particular, via the function call (you choose γ, or leave it empty)

```
[Fy_pos,CL,GAM,INFO] = hinfsyn(Ge, 1,1, gamma)
Fy = - Fy_pos
```

The optimal $\mathcal{H}_\infty$ regulator corresponds to the transfer function (after simplifications)

$$F_y = \frac{0.1084s^3 + 0.7591s^2 + 1.627s + 0.976}{s^4 + 9.136s^3 + 30.21s^2 + 42.52s + 21.28}$$

and $\gamma = 0.4949$.

(c) $S(i\omega)$ $T(i\omega)$ and $W_u(i\omega)$ are shown in Fig. 2, together with the corresponding bound $\gamma/W_*(\omega)$. Only $S(i\omega)$ is critical in the $\mathcal{H}_\infty$ design.

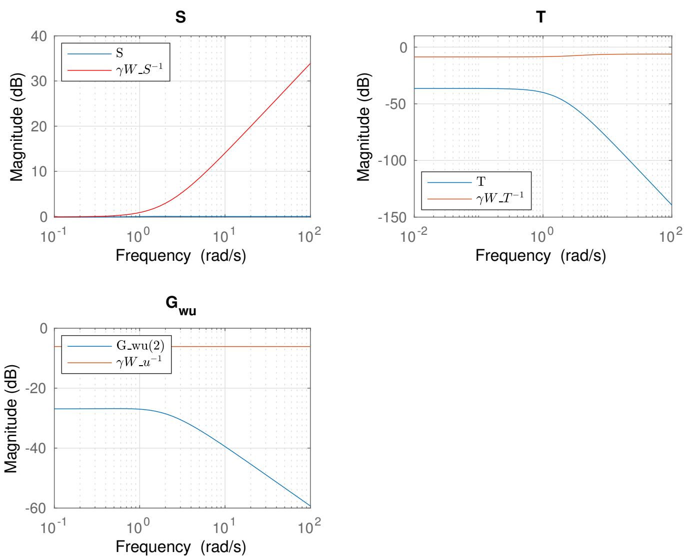

Figure 2: $\mathcal{H}_\infty$ design. In blue are S, T, and G_{wu}. In red are $\gamma W_S^{-1}(\omega)$, $\gamma W_T^{-1}(\omega)$, $\gamma W_u^{-1}(\omega)$.

5. (a) The Nyquist plot of the loop gain $KG(i\omega)$ is shown in Fig. 3, left, for different values of K. It shows that the loop gain crosses the critical point -1 when $K \approx 8$, hence the closed loop is stable for $K < 8$.

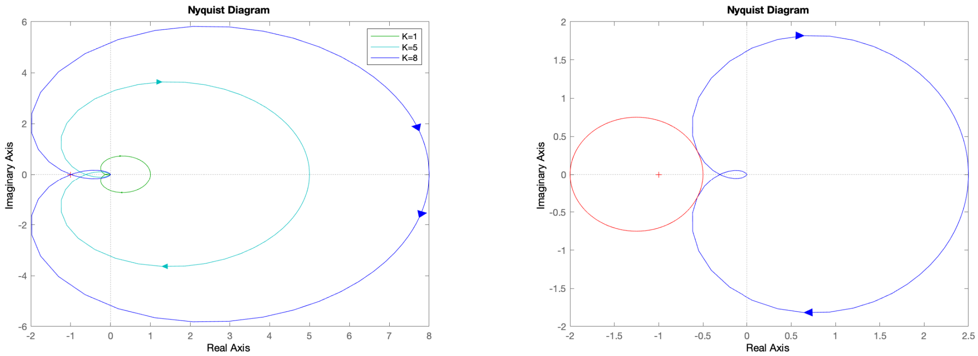

Figure 3: Exercise 5, Left: Nyquist plot of KG for different K. Right: circle criterion.

(b) Since the sector nonlinearity is upper bounded by k_2, the small gain theorem is applicable if $K \cdot k_2 \cdot \|G\|_\infty < 1$. Since $\|G\|_\infty = 1$ and $k_2 = 2$, it must be $2K < 1$, i.e., $K < 1/2$.

The full information for the sector can be used in the circle criterion. As shown in Fig. 3, right, The “forbidden” circle is centered at -1.25 and intersects the real axis at $-1/k_1 = -2$ and $-1/k_2 = -0.5$. For $K < 2.5$ the Nyquist curve does not intersect the circle.

The circle criterion is obviously less conservative than the small gain theorem, as the two values of K show. The reason is that it uses a more precise description of the nonlinearity.