Solution for TSRT09 Control Theory, 2021-06-08

1. (a) Non-minimum phase systems have a lower phase margin than their minimum phase counterparts (hence they are more stability-critical), and cannot be feedback regulated arbitrarily fast, i.e., the closed-loop bandwidth must be upper bounded.

   (b) The least common denominator of all minors is $p(s) = (s+1)(s+3)(s+7)$, hence the poles are $s = -1$, $s = -3$ and $s = -7$.

   (c) If v is a white noise of variance 1, then $\Phi_v(\omega) = 1$, hence when
$$y(s) = H(s)v(s) = \frac{1}{s+2}v(s)$$
   it is
$$\Phi_y(\omega) = |H(i\omega)|^2\Phi_v(\omega) = H(i\omega)H(-i\omega)\cdot 1 = \frac{1}{\omega^2+4}$$

   (d) When e.g. $d = 3$, then for $s \to \infty$ it is $y = u + 2$ meaning that no u s.t. $|u| \leq 1$ can completely cancel the effect of the disturbance.

   (e) For a physical system it is reasonable to assume that both plant and controller decay at least as fast as $1/\omega$, hence the Bode integral theorem can be applied. When the area A is bigger than the area B, then this theorem says that the loop gain must have at least one unstable pole.

2. (a) The pole polynomial is $p(s) = (s+1)^2(s-2)$, and the maximal minor (with $p(s)$ as denominator) is
$$\det G(s) = \frac{3(s+2-\alpha)}{p(s)}$$
   from which we see that the zero is $z = -2+\alpha$. When $\alpha < 2$ the system is minimum phase, and when $\alpha > 2$ it is non-minimum phase.

   (b) Exact dynamical decoupling requires to cancel the zero in $z = -2 + \alpha$. However, when $\alpha > 2$ then this requires to use $F(s)$ with an unstable pole, which is a bad idea (Never use unstable regulators!).

   (c) Steady-state decoupling corresponds to choosing a constant F such that $G(0)F$ is diagonal. For instance if $F = G^{-1}(0)$ then $\tilde{G}(0)$ is the identity matrix. Doing the calculations:
$$F = \frac{1}{2-\alpha}\begin{bmatrix} 2 & -1 \\ -2\alpha/3 & 2/3 \end{bmatrix}$$
   Notice that the term $2 - \alpha$ appearing in the denominator is ill-posed only in the case $\alpha = 2$ (rather that $\alpha \leq 2$, as it was in (b)).

3. (a) Since the penalty is quadratic we can use an LQ design to get $u = -Lx$ where L is computed with the `lqr` function in matlab:

$$L = \begin{bmatrix} 1.7321 & 4.1492 & 4.9697 & 3.1527 \end{bmatrix}$$

The poles are in this case

```
eig(A-B*L) =
  -0.5963 + 0.9801i
  -0.5963 - 0.9801i
  -0.9801 + 0.5963i
  -0.9801 - 0.5963i
```

(b) The output equation for the system is

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and the Kalman filter is

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

which, in order to get the full regulator, must be coupled with $u = -L\hat{x}$, where L is the one given above. The computation of K can be done using the `lqe` function of matlab (for each value of ρ) and the corresponding poles of the Kalman filter are the eigenvalues of $A - KC$. For $\rho = 10$ it is

$$K = \begin{bmatrix} 2.5680 & 0.1970 \\ 3.3166 & 0.9939 \\ 0.1970 & 2.5071 \\ 0.0058 & 3.1623 \end{bmatrix}$$

which corresponds to

```
eig(A-K*C) =
  -1.1714 + 1.3661i
  -1.1714 - 1.3661i
  -1.3661 + 1.1714i
  -1.3661 - 1.1714i
```

All real parts are more negative than those in (a) hence the poles of the Kalman filter are faster than those of the LQ regulator, which is normally a good rule of thumb.

(c) The LQ regulator in (a) is robust (e.g. infinite amplitude margin) while the LQG in (b) is not, see Sect. 9.4 in the book.

4. (a) When $u = 0$ the equilibria for the system are

$$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The Jacobian linearization for the system is

$$A = \begin{bmatrix} x_2 & -1 + x_1 \\ 2 + 3x_2 & -1 + 3x_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Computed at x_i^* it leads to

$$A_1 = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \quad \Rightarrow \quad \text{eig}(A_1) = \{-0.5 \pm 1.3229i\} \quad \Rightarrow \quad x_1^* \text{ is an as. stable focus}$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \text{eig}(A_2) = \{-1, 2\} \quad \Rightarrow \quad x_2^* \text{ is a saddle point}$$

(b) The linearized system at x_2^* is (A_2, B, C) (with $C = I$). Since it is controllable, pole placement can be done arbitrarily. For instance using an LQR design with $Q_1 = Q_2 = I$, one gets

$$L = \begin{bmatrix} 0.7305 & -1.0730 \\ -1.0730 & 3.9618 \end{bmatrix} \quad \Rightarrow \quad \text{eig}(A_2 - BL) = \{-1.5434, -2.1490\}$$

(c) The linearized system for the stable equilibrium is (A_1, B, C). Compute $G(s) = C(sI - A_1)^{-1}B$ and observe that $\text{RGA}(G(0)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence the pairing is $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$.

5. (a) The describing function $Y_f(C)$ for the saturation is given in Example 14.5 of the book, and it is always real (and negative), see blue line in Fig. 1. Since $r = 0$, the linear part of the system can be written as

$$\tilde{G}(S) = G(S)F(S) = \frac{K(\tau s + 1)}{\tau s(s + 2)^2}$$

whose Nyquist curve is given in Fig. 1 for 2 values of K (red and green curves). When K grows there is an intersection between $-1/Y_f(C)$ and $\tilde{G}(i\omega)$. Numerically, the intersection starts to appear when $K \sim 2.67$, and keeps being present when K grows, meaning that the prediction of the describing function method is that self-sustained oscillations are present for $K > 2.67$.

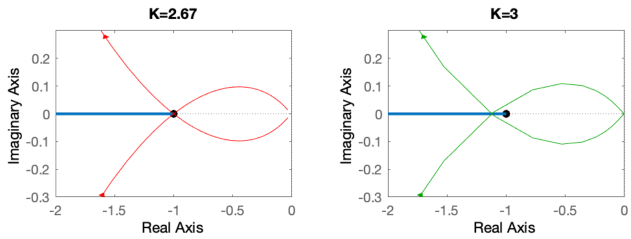

Figure 1: Describing function $-\frac{1}{Y_f(C)}$ and Nyquist curve of the loop gain $\tilde{G}(i\omega)$ for two values of K.

(b) The frequency of the oscillations can be obtained by computing the value of ω for which $\tilde{G}(i\omega)$ is real (since $-1/Y_f(C)$ is real). Doing the calculations:

$$\tilde{G}(i\omega) = \frac{10K(i\omega/10+1)}{i\omega(i\omega+2)^2} = \frac{10K\left((-4\omega+0.1\omega(4-\omega^2)) - i(4-0.6\omega^2)\right)}{\omega(16\omega^2-(4-\omega)^2)}$$

This expression is real when the imaginary term vanishes, i.e., for $\omega = \sqrt{20/3} = 2.582$ (note that the fact that this value of ω is similar to the value of K at which oscillations start is just a coincidence). The frequency does not change with K.

(c) Since $-1/Y_f(C)$ grows by moving from right to left, as in Fig. 14.9(a) of the book, the oscillations are stable.

(d) Simulating the system (from nonzero initial conditions), stable self-sustaining oscillations can indeed be seen, see Fig. 2. Notice that oscillations seem to start for slightly slower values of K than those predicted by the describing function method (which is an approximation), although as $t \to \infty$ they probably have $C \to 0$ when $K < 2.67$.

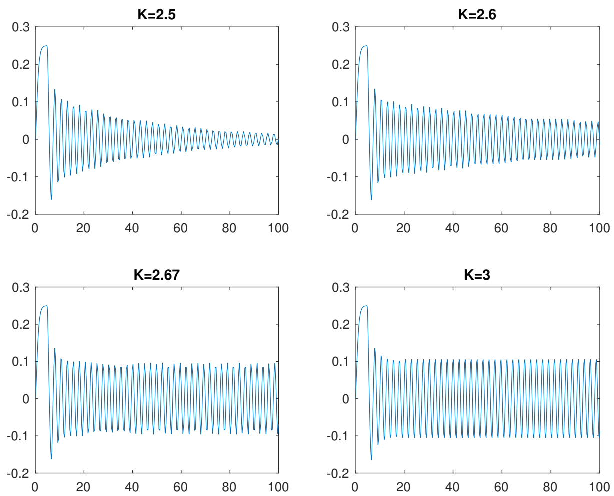

Figure 2: Oscillations as K is varied.