Solution for TSRT09 Control Theory, 2021-03-26

NOTE: in exercises 1, 3 and 4, the text of the exam had slightly different versions for different students. Here only one solution is presented but the other solutions are mentioned in "Alternative version".

1. 1-C: The Kalman filter is a state observer.

   2-D: This request is impossible (at least for $GF_y \sim s^{-2}$), as the Bode integral theorem states.

   3-E: LQ is optimal for a quadratic cost.

   4-A: The Lyapunov function method gives a sufficient condition for (global) asymptotic stability.

   5-B: The solution of an $\mathcal{H}_\infty$ design allows to exactly satisfy constraints on S and T.

   Alternative versions: different people got different orders of the answers.

2. (a) The origin is an equilibrium point, and the corresponding Jacobian linearization is

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$$

   of eigenvalues $\lambda_{1,2} = 0.5000 \pm 2.3979i$. Hence the origin is an unstable focus, meaning that both the linearization and (locally, near the origin) also the nonlinear systems are unstable.

   (b) We use the exact linearization method. Differentiating the output

$$\dot{y} = \alpha \dot{x}_1 + \beta \dot{x}_2 = \alpha \sin x_1 + \alpha x_2 + 2\beta x_1 + (2\alpha - \beta)u$$

   hence if $\beta = 2\alpha$ the term in u vanishes and we can keep differentiating:

$$\ddot{y} = \underbrace{\alpha\left((\cos x_1 + 4)\sin x_1 + x_2(\cos x_1 + 4) + 2x_1\right)}_{L_f^2 h(x)} + \underbrace{\alpha\left(2\cos x_1 + 7\right)}_{L_g L_f h(x)} u$$

   Now the feedback linearizing change of input is obtained as

$$u = \frac{-L_f^2 h(x) + v}{LgL_f h(x)} \qquad \Longrightarrow \qquad \ddot{y} = v$$

   which is a second order system when in state space representation.

   (c) i. The specifications are $|S(i\omega)| \leq 10^{-2}$ for $0 \leq \omega \leq 1$ and $|T(i\omega)| \leq 10^{-2}$ for $\omega \geq a$.

ii. If we use the approximations of eq. (7.12) and (7.13) in the book, then it must be $|GF_y| > 10^2$ for $0 \leq \omega \leq 1$ and $|GF_y| \leq 10^{-2}$ for $\omega \geq a$.

iii. The difficulty comes from the Bode relationship, which puts a bound on how fast $|GF_y|$ can change. If the slope of this amplitude is too big, stability can be compromised because of the amplitude-phase dependency given by the Bode relationship.

3. The plant is

$$G(s) = \begin{bmatrix} \frac{s+2}{(s-1)(s+1)} & \frac{s+4}{(s+2)(s+3)} \end{bmatrix}$$

(a) The pole polynomial is

$$p(s) = (s-1)(s+1)(s+2)(s+3)$$

and the poles are its roots: $p_i = +1, -1, -2, -3$.

(b) Rewrite $G(s)$ so that $p(s)$ is the common denominator

$$G(s) = \begin{bmatrix} \frac{(s+2)^2(s+3)}{p(s)} & \frac{(s-1)(s+1)(s+4)}{p(s)} \end{bmatrix}$$

The greatest common divisor of the numerators is 1, hence the system has no zero.

(c) The singular values are the square roots of the eigenvalues of $G^*(i\omega)G(i\omega)$. There are two of them, but since $G^*(i\omega)G(i\omega)$ has rank 1, only one singular value is $\neq 0$. It is plotted in Fig. 1.

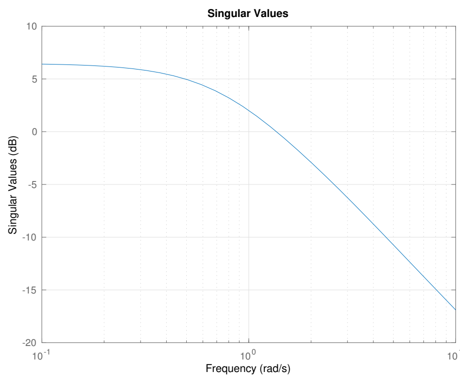

Figure 1: Exercise 3, singular value $\sigma(G)$.

(d) $\|G\|_\infty = 2.1082$, or $20\log_{10}(\|G\|_\infty) = 6.4782$ (i.e., the DC value in Fig. 1).

(e) $u_1 \leftrightarrow y$, since the first component of G is unstable.

Alternative version # 2 : The plant is

$$G(s) = \begin{bmatrix} \frac{s+4}{(s+2)(s+3)} & \frac{s+2}{(s-1)(s+1)} \end{bmatrix}$$

which is the same as the one above, with components exchanged. The solution is obviously the same, only with $u_2 \leftrightarrow y$ in the last item.

Alternative version # 3 : The plant is

$$G(s) = \begin{bmatrix} \frac{s+3}{(s-1)(s+1)} & \frac{s+3}{(s+2)(s+4)} \end{bmatrix}$$

- poles: $p_i = +1,\ -1,\ -2,\ -4$
- zeros: $z_i = -3$
- singular values: similar to Fig. 1
- $\|G\|_\infty = 3.0233$
- $u_1 \leftrightarrow y$

4. The plant is

$$G(s) = \begin{bmatrix} \frac{s+2}{(s-1)(s+1)} & \frac{s+4}{(s+2)(s+3)} \end{bmatrix}$$

(a) A PI regulator is given by

$$F_i = K_i \left(1 + \frac{1}{\tau_i s}\right) \qquad i = 1,\ 2$$

and the full regulator is therefore

$$F_y = [F_1 \ \ F_2]^T$$

Taking e.g. $K_1 = 10$, $K_2 = 1$ and $\tau_i = 1$, $i = 1,\ 2$, is enough to get closed loop stability, with a closed-loop transfer function

$$G_c = \frac{11(s + 3.358)(s^2 + 3.369s + 3.141)}{(s + 7.296)(s + 4)(s^2 + 3.704s + 3.975)}$$

The step response and the corresponding values of $|S|$ and $|T|$ are in Fig. 2. As expected, S is damped at low freq. and T at high freq. For the choice of control

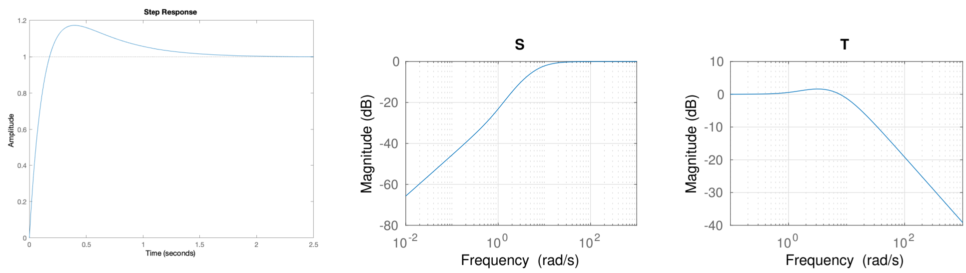

Figure 2: Exercise 4, PI design

parameters K_i and τ_i made above, the 4 transfer functions S, S_u, G_{wu} and G_{w_uy} are all stable, hence internal stability holds.

(b) Compute a minimal realization (A, B, C) of $G(s)$. Since we have a scalar output but 4 states, we need to set up an observer

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ u &- L\hat{x}\end{aligned}$$

where K is the solution of the observer problem `leq(A,B,C,R1,R2,R12)` (with $R_1 = I_2$, $R_2 = 1$ and $R_{12} = 0$) and L is the solution of the lqr problem `lqr(A,B,Q1,Q2)` (with $Q_1 = I_4$ and $Q_2 = I_2$).

$$K = \begin{bmatrix} 2.5695 \\ 1.6132 \\ 0.2605 \\ -0.0755 \end{bmatrix} \qquad L = \begin{bmatrix} 1.6180 & 1.6180 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.3028 & 0.3028 \end{bmatrix}$$

This leads to the transfer function $u \to y$

$$F_y = L(sI - A + BL + KC)^{-1}K$$

(of dimension 64!), from which we can calculate S and T, see Fig. 3. Notice that

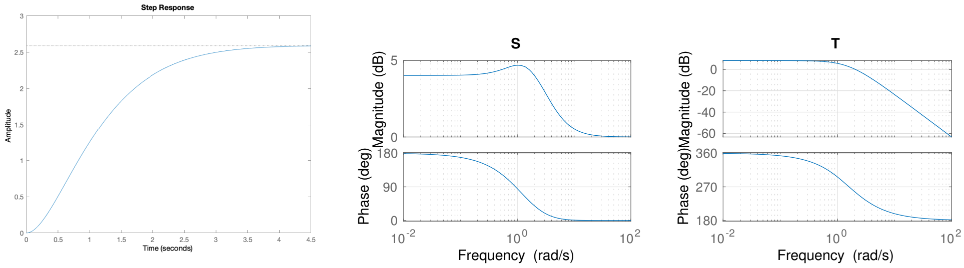

Figure 3: Exercise 4, LQG design

both $|S|$ and $|T|$ look large at low frequency, which seems at odds with $S + T = 1$. However, if you look at the phases, they are shifted by 180 degrees, meaning that the magnitudes are "off-phase" with each other (hence $S + T = 1$ is indeed valid). This is an example of lack of robustness in an observer-based LQG design.

(c) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \frac{s(s+2)}{(s+5)(s-1)(s+1)} & \frac{s(s+4)}{(s+3)(s+2)(s+5)} \\ \frac{1}{s+5} & \frac{(s+2)}{(s+5)(s-1)(s+1)} & \frac{(s+4)}{(s+3)(s+2)(s+5)} \\ 1 & \frac{s+2}{(s+1)(s-1)} & \frac{(s+4)}{(s+3)(s+2)} \end{bmatrix}$$

The matlab command for the $\mathcal{H}_2$ regulator is `[Fy_h2,Gc_h2,gamma,INFO] = h2syn(Ge,1,2)` and gives the value of $\mathcal{H}_2$ norm $\gamma = 0.53$. The controller gain (look inside `INFO`) is

$$L = 10^6 \begin{bmatrix} -3.3410 & -3.3729 & 0.2594 & -0.0267 & -5.1660 & 0.6034 \\ 3.3410 & 3.3729 & -0.2594 & 0.0267 & 5.1660 & -0.6034 \end{bmatrix}$$

and the observer gain (the system is not in innovation form, hence $K \neq N$)

$$K = \begin{bmatrix} 0.9820 \\ -1.2243 \\ 0.9031 \\ 2.5266 \\ 0.5730 \\ -1.2945 \end{bmatrix}$$

The resulting regulator is

$$F_y = \frac{2.7 \cdot 10^6 (s + 4.145)(s + 3)(s + 2)(s + 1)}{(s + 4.475)(s + 2.223)(s + 1.192)(s^2 + 4586s + 1.07 \cdot 10^7)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

S, T and G_{wu} are in Fig. 4.

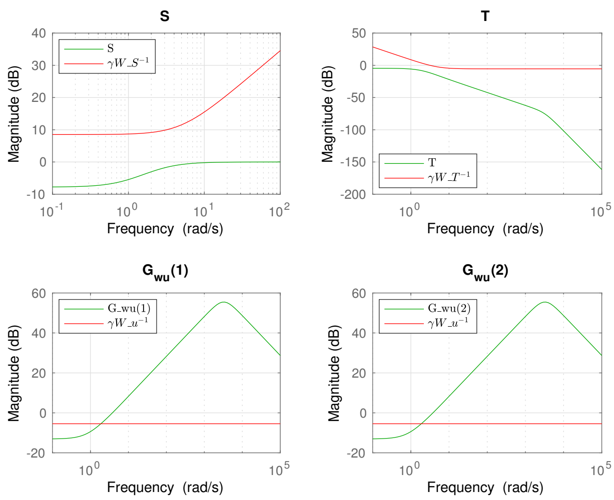

Figure 4: Exercise 4, $\mathcal{H}_2$ design

Alternative version # 2 : The plant is

$$G(s) = \begin{bmatrix} \frac{s+4}{(s+2)(s+3)} & \frac{s+2}{(s-1)(s+1)} \end{bmatrix}$$

which is the same as the one above, with components exchanged. The solution is obviously the same.

Alternative version # 3 : The plant is

$$G(s) = \begin{bmatrix} \frac{s+3}{(s-1)(s+1)} & \frac{s+3}{(s+2)(s+4)} \end{bmatrix}$$

and the solution is qualitatively similar to the previous ones.

5. (a) When we disregard the nonlinearity the system is always stable, for all K.

   (b) The nonlinearity $f(\cdot)$ is a sector nonlinearity:

$$0 = k_1 x \leq f(x) \leq k_2 x = x$$

   i.e., we can use the circle criterion with $k_1 = 0$ and $k_2 = 1$. This gives the violet stripe shown in Fig. 5 (“infinite circle” passing through the point $-1/k_2 = -1$). Since G (in green) is completely to the right of this “circle”, the system is stable.

   (c) In order to apply the small gain theorem, it must be $\|f\|\|G\|_\infty < 1$. Here $\|f\| = k_2 = 1$ and $\|G\|_\infty = \infty$ since G has a pole in the origin (the Nyquist diagram goes to ∞ when $\omega \to 0^+$, see Fig. 5). Hence it is not possible to use the small gain theorem.

   (d) For the given $G(i\omega)$, the Nyquist plot becomes parallel to the imaginary axis when $\omega \to 0^+$, with an asymptotic real part equal to $K/2^2 = K/4$. Hence the Nyquist plot is to the right of the “forbidden circle” if $K < 4$, see Fig. 5. Even varying K, the small gain theorem can never be applied.

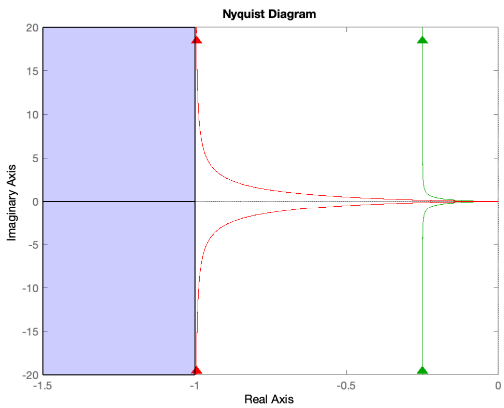

Figure 5: Exercise 5. Nyquist plot of the loop gain $KG(i\omega)$ for $K = 1$ (green) and $K = 3.99$ (red)