Solution for TSRT09 Control Theory, 2020-08-25

1. (a) The requirement is that the pair (A, B) is controllable.

   (b) A pole in the r.h.p. worsen the sensitivity function. If the pole is in p then the bandwidth must be at least $2p$.

   (c) It is not possible to move a zero from the left half plane to the right half plane through stable feedback. To see it, consider a feedback like $u = -F_y(s)(r - y)$. This gives the closed loop system

   $$G_c = \frac{GF_y}{1 + GF_y}$$

   which has the same zeros as GF_y. If you are required to use an internally stable regulator F_y, then it is not possible to use F_y to cancel the r.h.s. zero of G. G_c must therefore have all r.h.p. zeros of G.

   (d) The minors are $\frac{1}{s+1}$, $\frac{\beta}{s+1}$ and $\frac{1-\beta}{(s+1)^2}$. If $\beta = 1$, then the pole in -1 has multiplicity 1, if $\beta \neq 1$ it has multiplicity 2.

2. (a) The state space realization is

   $$\begin{aligned} \dot{x} &= -x + v_1 \\ y &= x + v_2 \end{aligned} \tag{1}$$

   The Kalman gain is given by $K = (P + 1)$ where P solves the ARE

   $$-P - P + 6 - (P + 1)^2 = 0 \quad \Longrightarrow \quad P = 1 \quad \Longrightarrow \quad K = 2$$

   Writing the Kalman filter:

   $$\dot{\hat{x}} = -\hat{x} + K(y - \hat{x}) = -3\hat{x} + 2y \tag{2}$$

   (b) From the expression (2), the TF is straightforwardly computed as $G_{y\hat{x}} = \frac{2}{s+3}$

   (c) From (1), the stationary variance of x can be computed as the solution Π_x of the Lyapunov equation

   $$A\Pi_x + \Pi_x A^T + NR_1N^T = -2\Pi_x + 6 = 0$$

   i.e., $\Pi_x = 3$. Rewriting (1) as $X(s) = \frac{1}{s+1}V_1(s)$, the spectral density of x is

   $$\Phi_x(\omega) = R_1 \frac{1}{(1 + i\omega)(1 - i\omega)} = \frac{1}{1 + \omega^2}$$

   (d) Recall that for the Kalman filter $\nu = y - C\hat{x} = y - \hat{x}$ is a white noise (innovation) of intensity $R_2 = 1$. Hence the observer can also be written as

   $$\dot{\hat{x}} = -\hat{x} + K\nu, \qquad \nu = \text{white noise of intensity } 1$$

Using this system, the stationary variance of $\hat{x}$ is given by the solution $\Pi_{\hat{x}}$ of the Lyapunov equation

$$-\Pi_{\hat{x}} - \Pi_{\hat{x}} + 4 = 0 \quad \Longrightarrow \quad \Pi_{\hat{x}} = 2$$

Since the corresponding transfer function is $G_{\nu\hat{x}} = \frac{2}{s+1}$, and since $A = -1$ is stable, the spectrum of $\hat{x}$ is

$$\Phi_{\hat{x}}(\omega) = R_2 G_{\nu\hat{x}} G_{\nu\hat{x}}^* = \frac{4}{(1+i\omega)(1-i\omega)} = \frac{4}{1+\omega^2}$$

(e) Comparing Π_x and $\Pi_{\hat{x}}$, it is indeed

$$\Pi_x - \Pi_{\hat{x}} = 3 - 2 = E\tilde{x}^2 = P = 1$$

3. RGA exercise

(a) We should not pair together signals that have a negative value in RGA at frequency 0.

$$\text{RGA}(G(0)) = \begin{bmatrix} -\frac{1}{14} & \frac{15}{14} \\ \frac{15}{14} & -\frac{1}{14} \end{bmatrix}$$

The best possible pairing is $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$.

(b) Choosing the P-regulator

$$F(s) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

the close-loop system is computed as `Gc=minreal(feedback(G*F,eye(2)))` and its poles are $s = -25.5386$, s=-17.4510, $s = -4.0365$ and $s = -0.9739$.

(c) The sensitivity function is given by `S=minreal(feedback(eye(2),G*F))` The singular values of S are shown in Fig. 1. If we look at the largest singular value,

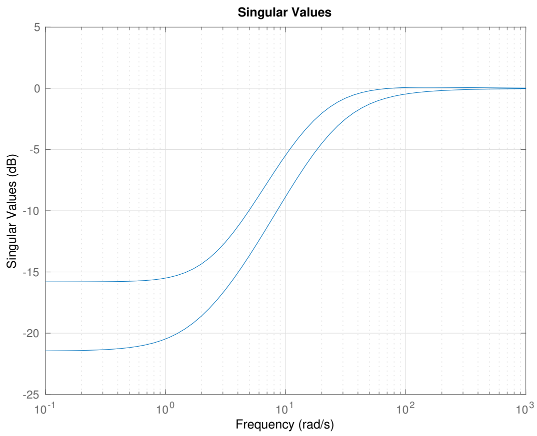

Figure 1: Exercise 3

at frequency 0 it is 0.162 (i.e., -15.8 dB) meaning that a constant disturbance is attenuated around 6.17 times. For S, the the crossover frequency (sensitivity crossover: where the largest singular value passed from damping to amplification) is around $\omega_c = 73$ rad/sec. Hence disturbances are damped below ω_c.

4. Describing function exercise.

   (a) The describing function for a relay with deadzone is

   $$Y_f = \frac{4}{\pi C}\sqrt{1-\frac{1}{C^2}}, \quad C > 1$$

   which is always real. The graph of $-\frac{1}{Y_f(C)}$ also belongs to the negative real axis. The describing function begins at $-\infty$ for $C = 1$, then moves towards the origin and then again towards $-\infty$. The nearest point to the origin corresponds to the solution of

   $$C^* = \arg\max_C Y_f(C) = \arg\max_C \frac{1}{C^2}(1-\frac{1}{C^2})$$

   which gives $C^* = \sqrt{2}$ and corresponds to $-\frac{1}{Y_f(C^*)} = -\frac{\pi}{2}$. The Nyquist curve of the linear system

   $$G(i\omega) = \frac{K}{i\omega(i\omega+1)(i\omega+2)}$$

   hits the negative real axis when $\omega_0 = \sqrt{2}$ and assumes there the value $|G(i\omega_0)| = \frac{K}{6}$. The system therefore will have no sustained oscillations if

   $$-\frac{\pi}{2} < -\frac{K}{6} \implies \frac{K}{6} < \frac{\pi}{2} \implies K < 3\pi$$

   The solution can also be computed graphically by plotting $-\frac{1}{Y_f(C)}$.

   (b) If the allowed amplitude for the oscillations at e is 3, then it means that $C = 3$. We can compute the corresponding value of K by solving the equation

   $$-\frac{1}{Y_f(3)} = G(i\sqrt{2})$$

   which directly gives $K = \frac{27\pi}{4\sqrt{2}}$. The angular frequency of the oscillations, according to (a), is $\omega_0 = \sqrt{2}$.

5. Phase plane exercise.

   (a) The origin is an equilibrium point for the system. The feedback gain L must be chosen so that the eigenvalues of the closed loop system have negative real part. One possible choice is for instance to place them at -1 and -2 with the following Matlab commands

   ```
   >> A = [0 1;0 -1];
   >> B = [0 1]';
   >> L = place(A,B,[-1 -2])
   L =
         2      2
   ```

   The closed loop system matrix $A - BL = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ has eigenvalues and eigenvectors:

```
>> [v,d] = eig([0 1;-2 -3])
v =
    0.7071   -0.4472
   -0.7071    0.8944
d =
    -1      0
     0     -2
```

The slower mode -1 has eigenvector $(1,-1)^T$ while the faster mode -2 has eigenvector $(-1,2)^T$. The phase plane is shown in Fig. 2.

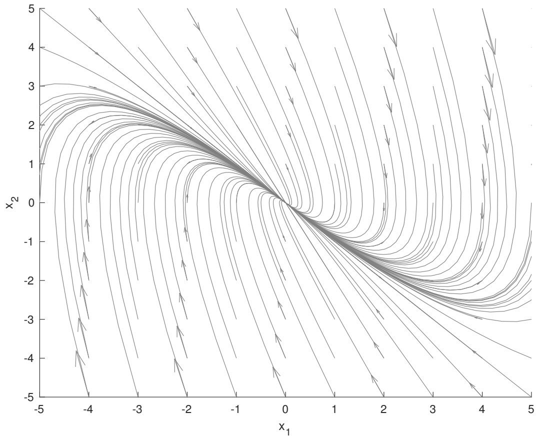

Figure 2: Exercise 5

(b) Using the feedback gain computed in (a) and the saturation function, we have the three cases:

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_2 + 1 \end{cases} \qquad \text{for } x_1 + x_2 < -1/2$$

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_1 - 3x_2 \end{cases} \qquad \text{for } |x_1 + x_2| < 1/2$$

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_2 - 1 \end{cases} \qquad \text{for } x_1 + x_2 > 1/2$$

In the region $|x_1 + x_2| < 1/2$ we have the same phase portrait as in (a). So in particular, the behavior around the origin is identical to the one in (a), meaning that we still have a stable equilibrium in the origin. Away from the origin (or more precisely, outside the band $|x_1 + x_2| < 1/2$), the system behave somewhat differently. If we compute $\frac{\dot{x_2}}{\dot{x_1}} = -1 \pm \frac{1}{x_2}$, then we see that for large values of x_2 (both positive and negative) $\frac{\dot{x_2}}{\dot{x_1}} \simeq -1$. It can also be noticed that for $x_2 = 1$ and $x_1 < -\frac{3}{2}$, x_2 is constant (i.e., $\dot{x}_2 = 0$), which corresponds to an horizontal line in the phase portrait. The same thing happens for $x_2 = -1$ and $x_1 > \frac{3}{2}$. The phase plane is shown in Fig. 3.

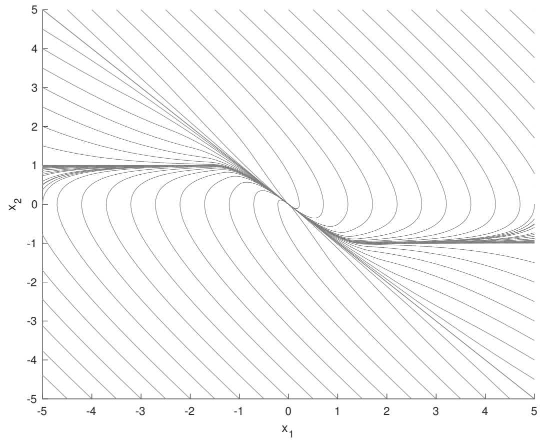

Figure 3: Exercise 5