Solution for TSRT09 Control Theory, 2020-06-09

1. (a) The Bode integral theorem puts a constraint on the sensitivity function S: it cannot be rendered small everywhere. If you compress it on a certain frequency range it will blow up on some other frequency range.

   (b) The system is not controllable, but we can use the PBH test to check stabilizability. The eigenvalues of A are $\lambda = \pm 1$. In correspondence of the $\lambda = 1$ (unstable eigenvalue)

   $$\begin{bmatrix} A - I & B \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

   has rank 2, while in correspondence of $\lambda = -1$ (stable)

   $$\begin{bmatrix} A + I & B \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   has rank 1. Hence the system is stabilizable (the uncontrollable mode is already stable). You can of course show the same thing by explicitly designing a state feedback.

   (c) RGA at steady state is ($\odot$ = element-wise multiplication)

   $$\text{RGA}(G(0)) = G(0) \odot (G^{-1}(0))^T = \begin{bmatrix} \frac{1}{2} & 10 \\ \frac{1}{5} & \frac{5}{3} \end{bmatrix} \odot \begin{bmatrix} -\frac{10}{7} & \frac{6}{35} \\ \frac{60}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{bmatrix}$$

   (d) Differentiate the output twice

   $$\begin{aligned} \dot{y} &= \dot{x}_1 = -2x_2 \\ \ddot{y} &= -2\dot{x}_2 = -2x_1 + 2x_2 - 2\tan x_1 + 2u \end{aligned}$$

   We want to choose u so that $\ddot{y} = v$ where v is the new input. This is obtained by choosing

   $$u = x_1 - x_2 + \tan x_1 + \frac{v}{2}$$

   (e) The noise n can be modeled as the output of a linear system H driven by a white noise of intensity 1:

   $$H(s) = \frac{1}{s+1}$$

   A state space representation of H is

   $$\dot{\tilde{x}}_2 = -\tilde{x}_2 + \tilde{n}$$

   Denoting $\tilde{x}_1 = x$, we obtain the augmented model

   $$\begin{aligned} \dot{\tilde{x}} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{n} \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \tilde{x} + u \end{aligned}$$

2. (a) A proper Q to be used in the ICM design is given by

$$Q(s) = \frac{G^{-1}(s)}{(s+1)^2} = \frac{s+1}{s+2}$$

In fact, this Q is stable and leads to the closed loop system

$$G_c(s) = G(s)Q(s) = \frac{1}{(s+1)^2}$$

which is of course also stable and has the poles in the same location as $G(s)$. The corresponding feedback is

$$F_y = (I - QG)^{-1}Q = \frac{(s+1)^3}{s(s+2)^2}$$

(b) The loop gain

$$F_y G = \frac{1}{s(s+2)}$$

has an integrator, which means that the steady state error in the step response is 0.

(c) Looking at the disturbance-related TFs of the system in Fig. 1, we can see that the best combination of disturbances for our control design is (ii), as both S and G_{wu} (dealing with w) are damped at low frequency, while T (dealing with n) and $G_{w_u y}$ (dealing with w_u) are damped at high frequency.

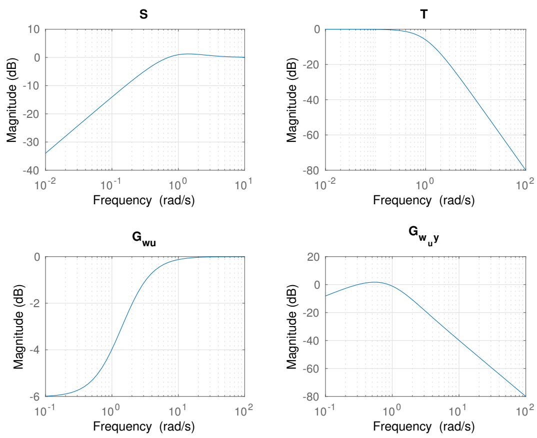

Figure 1: Exercise 2. S, T, G_{wu} and $G_{w_u y}$.

3. (a) The transfer function of the system is

$$G(s) = \begin{bmatrix} \frac{1}{s+0.4} & \frac{6(s+0.6)}{(s+2.414)(s+0.4)(s-0.4142)} \\ 0 & \frac{4(s-2)}{(s+2.414)(s-0.4142)} \end{bmatrix}$$

Notice that the system is unstable. The poles polynomial is

$$p(s) = (s + 0.4)(s - 0.4142)(s + 2.4242),$$

meaning that the poles of the system are $p_i = -2.4242, -0.4, 0.4142$. The only zero of G is $z = 2$ (in this case: pole of $G^{-1}(s)$, after simplifications).

(b) For a MIMO system, the static gain is computed from the singular values of $G(0)$: if `[V,D]=eig(G0'*G0)`, then $\sigma = \sqrt{\text{diag}(D)}$ has 2 components

$$\sigma_1 = 1.6409 \qquad \sigma_2 = 12.1884,$$

meaning that the static gain of $G(0)$ is in the interval $[\sigma_1, \sigma_2]$. The directions of u corresponding to the two boundaries σ_1 and σ_2 are given by the eigenvectors in V;

$$u_1 = V(:,1) = \begin{bmatrix} -0.9877 \\ -0.1562 \end{bmatrix}, \qquad u_2 = V(:,2) = \begin{bmatrix} -0.1562 \\ 0.9877 \end{bmatrix}$$

(c) The static decoupling matrix is

$$F = G(0)^{-1} = \begin{bmatrix} 0.4000 & 0.4500 \\ 0 & 0.1250 \end{bmatrix}$$

(d) LQ regulator. Since the problem requires feedback from the output, one need to build an observer. Since there is no disturbance in the model $N = 0$. Choosing for instance $R_1 = 0$ and $R_2 = I$, the solution to the LQ problem (Kalman gain) is

$$K = \begin{bmatrix} 0.2577 & -0.2187 \\ 0.2238 & -0.1900 \\ -0.1822 & 0.1547 \end{bmatrix}$$

while for the regulator gain it is, with $Q_1 = I$ and $Q_2 = I$,

$$L = \begin{bmatrix} -0.0352 & 0.6467 & -0.1269 \\ 0.7836 & 0.2538 & -1.2478 \end{bmatrix}$$

The full regulator is then

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ u &= -L\hat{x} \end{aligned}$$

(e) LQ regulators from reconstructed states are not robust, see Sect. 9.4 of the book.

4. (a) The extended system is

$$G_e = \begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ 1 & G \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{s+2}{(s+0.2)(s-1)} \\ \frac{1}{s+1} & \frac{s+2}{(s+0.2)(s-1)(s+1)} \\ 1 & \frac{s+2}{(s+0.2)(s-1)} \end{bmatrix}$$

which corresponds to the state space model (after elimination of redundant zero/poles with `minreal`)

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \\ y &= Cx + w \end{aligned}$$

with

$$A = \begin{bmatrix} 0.704 & 0.5443 & -0.5512 \\ 0.4507 & 0.2174 & -0.02642 \\ 0.2423 & -0.309 & -1.121 \end{bmatrix}, \quad B = \begin{bmatrix} 1.746 \\ 2.978 \\ -0.2956 \end{bmatrix}, \quad N = \begin{bmatrix} 0.2376 \\ -0.0736 \\ 0.6619 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1.064 & -0.3296 & -0.4188 \\ 0.4753 & -0.1472 & 1.324 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1.064 & -0.3296 & -0.4188 \end{bmatrix}$$

The condition $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is satisfied, but $A - NC$ is not stable, hence the model is not in innovation form (i.e., N is not the Kalman gain).

(b) An $\mathcal{H}_\infty$ controller can be designed using the function **hinfsyn** in matlab. In particular, via the function call (you choose γ, or leave it empty)

```
[Fy_pos,CL,GAM,INFO] = hinfsyn(Ge, 1,1, gamma)
Fy = - Fy_pos
```

The optimal $\mathcal{H}_\infty$ regulator corresponds to the transfer function (after simplifications)

$$F_y = \frac{415.56(s+0.2)}{(s+216.8)(s+1.664)}$$

(c) When called with $\gamma = 1$, the **hinfsyn** function returns no controller, hence an $\mathcal{H}_\infty$ solution does not exist. The $\mathcal{H}_\infty$ norm is in fact $\gamma = 1.9343$.

(d) From Fig. 2, $S(\omega)$ is never critical, while both $T(\omega)$ and $W_u(\omega)$ approach the corresponding upper bound (in red). Computing the $\mathcal{H}_\infty$ norms, it is $\|T\|_\infty = 1.77$ and $\|G_{wu}\|_\infty = 1.924$ hence G_{wu} is the most critical constrain in this $\mathcal{H}_\infty$ control design.

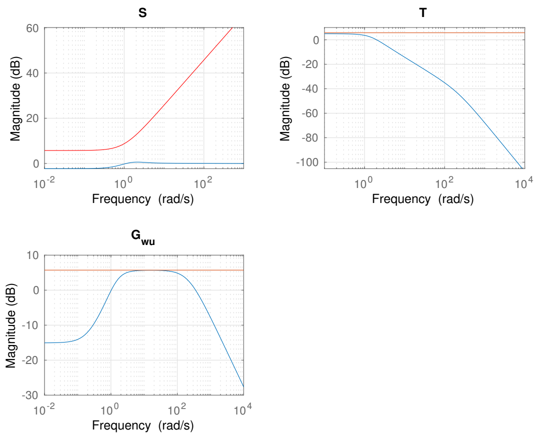

Figure 2: $\mathcal{H}_\infty$ design. In blue are S, T, and G_{wu}. In red are $\gamma W_S^{-1}(\omega)$, $\gamma W_T^{-1}(\omega)$, $\gamma W_u^{-1}(\omega)$.

5. (a) The gain for the nonlinearity is 1. The gain for $G(s)$ can be computed e.g. looking at the Bode diagram and observing that it is the DC gain: $\|G\|_\infty = 20/3 = 6.67$. Hence, from the small gain theorem,

$$K\|f\|_2\|G\|_\infty < 1 \quad \iff \quad K < \frac{1}{\|f\|_2\|G\|_\infty} = K_1 = 3/20$$

(b) For the saturation function, the two slopes describing the nonlinearity in the circle criterion are $k_1 = 0$ and $k_2 = 1$. Hence the circle has an infinite radius and it is passing through the point -1 (see Fig. 3). In order for the Nyquist curve of G to be outside of this circle, one must compute the leftmost point of its Nyquist plot, i.e., $\min_s \mathrm{Re}(G(s)) = -1.4$ (see green curve in Fig. 3). The upper bound on K is obtained as $K_2 = 1/1.4 = 0.71$.

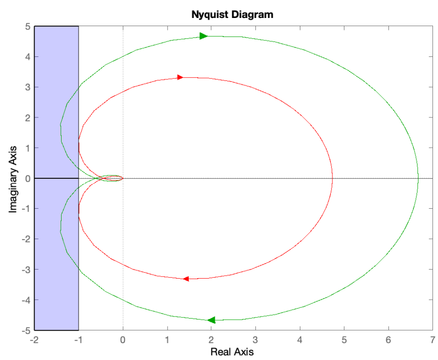

Figure 3: Computing upper bound on K using the circle criterion. Blue: the "forbidden" circle. Green: Nyquist plot of G. Red: Nyquist plot of $K_2G(s)$.

(c) The describing function of the saturation is real and $-\frac{1}{Y_f(C)}$ has the curve shown in blue in Fig. 4, i.e., it "ends" at -1 (the critical point in the Nyquist diagram).

Hence as long as $KG(S)$ does not cross or circle -1, both the open loop and the closed loop system are stable. Since the amplitude margin for $G(s)$ is 1.6, it must be $K < K_3 = 1.6$.

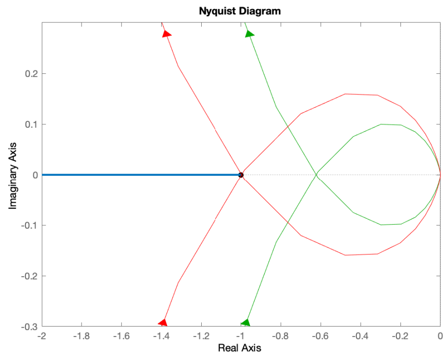

Figure 4: Computing upper bound on K via describing function. Blue: $-\frac{1}{Y_f(C)}$; Green: Nyquist plot of G; red: Nyquist plot of $K_3G(s)$.

(d) Of the 3 bounds, $K_1 = 0.15$ is the most conservative, then $K_2 = 0.71$ and finally the least conservative is $K_3 = 1.6$ (which is the exact value in this case).