Lecture 1. Rigid body motion.

Mikael Norrlöf, mino@isy.liu.se
Content

- Rigid body transformation
- Rotation
  - Rotation matrices
  - Euler’s theorem
  - Parameterization of SO(3)
- Homogeneous representation
  - Matrix representation
  - Chasles’ theorem
Background to modeling

Kinematics
• studies the motion of objects without consideration of the circumstances leading to the motion

Dynamics
• studies the relationship between the motion of objects and its causes
Rigid body motion
The motion of a rigid body can be parameterized as:

- position
- orientation

of one point of the object. The configuration.
Content

• Rigid body transformation
• Rotation
  – Rotation matrices
  – Euler’s theorem
  – Parameterization of SO(3)
• Homogeneous representation
  – Matrix representation
Representation of orientation

- Rotation matrices
- Angle – axis representation
- Euler angles
- Quaternion
- Exponential coordinates
- …
Composition of rotation

The order of rotation axes is important

\[ R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \]

\[ R_2(\alpha) = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \]

\[ R_3(\alpha) = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Rx – pi/2
Ry – pi/4

ii) xyz, iii) zyx
Example

Example 2.8

Suppose $R$ is defined by the following sequence of basic rotations in the order specified:

1. A rotation of $\theta$ about the current $x$-axis
2. A rotation of $\phi$ about the current $z$-axis
3. A rotation of $\alpha$ about the fixed $z$-axis
4. A rotation of $\beta$ about the current $y$-axis
5. A rotation of $\delta$ about the fixed $x$-axis

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x, \theta}$ and pre- or post-multiply as the case may be to obtain

$$R = R_{x, \delta} R_{z, \alpha} R_{x, \theta} R_{z, \phi} R_{y, \beta} \quad (2.24)$$
Euler angles
Euler angles

😊 Gimbal lock  (Apollo IMU Gimbal lock 1, 2)

\[
R(\alpha, \frac{\pi}{2}, \gamma) = \begin{pmatrix}
0 & \cos \gamma \sin \alpha - \cos \alpha \sin \gamma & \cos \alpha \cos \gamma + \sin \alpha \sin \gamma & 0 \\
0 & \cos \alpha \cos \gamma + \sin \alpha \sin \gamma & \cos \alpha \sin \gamma - \cos \gamma \sin \alpha & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

= \begin{pmatrix}
0 & \sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\
0 & \cos(\alpha - \gamma) & \sin(\alpha - \gamma) & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
Euler angles

😢 Implementing interpolation is difficult
😢 Ambiguous correspondence to rotations
😢 The result of composition is not apparent
😢 Non-linear dynamics

😊 Mathematics is well known
😊 Can be visualized “in the mind”
Quaternions

Sir William Rowan Hamilton (1809-1865)

Lectures on Quaternions: Containing a systematic statement of

A New Mathematical Method

Of which the principles were communicated in 1843 to the Royal Irish Academy; and which has since formed the subject of successive courses of lectures, delivered in 1848 and subsequent years in the halls of Trinity College, Dublin: with numerous illustrative diagrams, and with some geometrical and physical applications.
Quaternions

Generalization of complex numbers to 3D.

\[ s + i x + j y + k z \]

with \( i^2 = j^2 = k^2 = ijk = -1 \), \( ij = -ji = k \), \( jk = -kj = i \), \( ki = -ik = j \).

A quaternion is usually represented as \( q = <s, v> \) with

- \( s \) scalar (real part)
- \( v \) vector in \( \mathbb{R}^3 \) (complex part)

Unit quaternion \( ||q|| = 1 \).
Rotation with quaternions

Angle axis to quaternion

\[ \theta, v \Rightarrow q = \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} v \right) \]

Composition of rotations, \( q_1 \) then \( q_2 \)

\[ q = q_2 q_1 \]
Rotation with quaternions

Rotation of a vector, \( u = Rv \)

\( v_q = <0, v>, \) \( q \) is quaternion representation of \( R \)

\( u_q = qv_qq^{-1} = <0, u> \)

\[
R_q(q) = \begin{bmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\
2q_1q_2 - 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_0q_1 \\
2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{bmatrix}.
\]
Some remarks

• $q$ and $-q$ represent the same rotation

\[ q = <s,v> \text{ and } q^{-1} = <s,-v> \]
Quaternions

😍 Can only represent orientation
😊 Quaternion math is not so well known
😍 Compact representation, based upon
😊 Simple interpolation methods
😊 No gimbal lock
😊 Simple composition
😊 Linear (bi-linear) dynamics, (NASA)
Homogeneous transformations

\[
\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \text{Rot}_{x,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
Composition Rule for Homogeneous Transformations
The same interpretation regarding composition and ordering of transformations holds for $4 \times 4$ homogeneous transformations as for $3 \times 3$ rotations. Given a homogeneous transformation $H_1^0$ relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$
Comparison for different operations

### Performance comparison of rotation chaining operations

<table>
<thead>
<tr>
<th>Method</th>
<th>Storage</th>
<th># multiplies</th>
<th># add/subtracts</th>
<th>total operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation matrix</td>
<td>9</td>
<td>27</td>
<td>18</td>
<td>45</td>
</tr>
<tr>
<td>Quaternions</td>
<td>4</td>
<td>16</td>
<td>12</td>
<td>28</td>
</tr>
</tbody>
</table>

### Performance comparison of various rotation operations

<table>
<thead>
<tr>
<th>Method</th>
<th>Storage</th>
<th># multiplies</th>
<th># add/subtracts</th>
<th># sin/cos</th>
<th>total operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation matrix</td>
<td>9</td>
<td>9</td>
<td>6</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>Quaternions</td>
<td>4</td>
<td>21</td>
<td>18</td>
<td>0</td>
<td>39</td>
</tr>
<tr>
<td>Angle/axis</td>
<td>4*</td>
<td>23</td>
<td>16</td>
<td>2</td>
<td>41</td>
</tr>
</tbody>
</table>
Further studies

- R.M. Murray, Z. Li, and S.S. Sastry: *A mathematical introduction to Robotic Manipulation* (Chapter 2)
- James Diebel: *Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors*
- Erik B. Dam, Martin Koch, and Martin Lillholm: *Quaternions, Interpolation and Animation*