Stochastic systems

Continuous-time processes, SDE,Martingales

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Introduction

A stochastic process for which the index set T is an interval is called a *continuous time process*. The index set T is defined as $[0, \infty)$.

Counting process. Let $N(\cdot) = \{N(t), t \geq 0\}$ a stochastic process where $N(t)$ represents the number of various outcomes in a system over time. Here $N(t)$ is integer, and if $s \leq t$ then $N(s) \leq N(t)$.

Definition

Consider $X(\cdot) = \{X(t), t \geq 0\}$ a stochastic process . It is said that $X(\cdot)$ has
a) **independent increments** if for any $n \geq 1$ and for $0 \leq t_0 < t_1 < .. < t_n$ the increments

$$X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1}) \qquad (1)$$

are independent.
b) **stationary increments** if for any $t \geq 0$ and $h > 0$ the probability distribution of any increment $X(t+h) - X(t)$ depends only on h,i.e.,

$$X(t+h) - X(t) \sim X(s+h) - X(s) \forall s, t \geq 0$$

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Lévy process

A stochastic process $X(\cdot) = \{X(t), t \geq 0\}$ is a Lévy process if
a) $X(0) = 0$ a.s.

b) $X(\cdot)$ has independent and stationary increments, and

c) $X(\cdot)$ is a stochastic continuous process.

The most well known Lévy processes are the Poisson process and the Wiener process.

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Poisson process

Define a continuous-time counting process $N(\cdot) = \{N(t), t \geq 0\}$ and a positive number λ. The process $N(\cdot)$ is a Poisson process with a rate parameter λ if
a) $N(\cdot)$ has independent increments and
b) $N(\cdot)$ has stationary increments such that

$$N(t+h) - N(t) \sim Poi(\lambda h)\ \forall t \geq 0, h > 0$$

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Brownian Motion

A process $W(\cdot) = \{W(t), t \geq 0\}$ is a Wiener process, or a Brownian motion, if
a) $W(0) := 0$,
b) has independent increments, and
c) has stationary increments with

$$W(t+h) - W(t) \sim N(0, \sigma^2 h)\ \forall t \geq 0, h > 0$$

where σ is a positive constant. If $\sigma^2 = 1$, it is said that $W(\cdot)$ is a standard Wiener process.

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Stochastic differential equations

An ordinary differential equation $\frac{dx}{dt} = f(t,x)$ can be replaced by a random differential equation

$$\frac{dX}{dt} = F(t,X,Y)$$

where $Y = Y(t)$ represents some stochastic input process. In particular is not possible to interpret it as an ordinary differential equation along each path and the solution is a stochastic process. This happens when the differential equation has the form

$$\frac{dX}{dt} = f(t,X) + g(t,X)N$$

with N being a Gaussian white noise process.

The irregularity of the sample paths of N makes the equation intractable mathematically. A solution should be a solution of the random integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s,X(s))ds + \int_{t_0}^{t} g(s,X(s))N(s)ds \qquad (2)$$

since the last integral in (2) cannot be defined in any meaningful way, it is replaced by an integral of the form

$$\int_{t_0}^{t} g(s,X(s))dW(s)$$

where W is the Wiener process.

The impact here is that the integral inherits many of the probabilistic properties of the Wiener process and the corresponding calculus will differ from the Stieltjes case. The main instrument of this calculus is Ito´s formula, that will yield a stochastic equation for any sufficiently smooth function of a solution.

Ito's formula is the basis for any analysis of solutions of stochastic equations along the lines of the qualitatives approaches used in ordinary differential and integral equations.

General assumption. Let (Ω, F, P) be a complete probability space equipped with a filtration $\{F_t\}_t$ satisfying the usual conditions, and assume that on this space a Brownian motion $\{(W_t, F_t)\}_{t\in[0,\infty)}$ with respect to this filtration is defined.

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Ito's formula

Definition

Let $(W(t), F_t)_{t\in[0,\infty)}$ be an m-dimensional Brownian motion, $m \in \mathbb{N}$.
1) $(X(t), F_t)_{t\in[0,\infty)}$ is called a real-valued Ito process if for all $t \geq 0$ it admits the representation

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s)$$

$$= X(0) + \int_0^t f(s)ds + \sum_{j=1}^{m} \int_0^t g_j(s)dW_j(s) \; a.s. \, P.$$

2) An n-dimensional Ito process $X = (X^{(1)}, .., X^{(n)})$ consists of a vector with components being real-valued Ito processes.

One-dimensional Ito formula

Let W_t be a one-dimensional Brownian motion, and X_t a real-valued Ito process with

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s dW_s$$

Let $f: \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then, for all $t \geq 0$ we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle x\rangle_s$$

$$= f(X_0) + \int_0^t (f'(X_s)\cdot K_s + \tfrac{1}{2}\cdot f''(X_s)\cdot H_s^2)ds + \int_0^t f'(X_s)H_s dW_s \; a.s. \, P.$$

All integrals appearing above are defined.

Remark: The Ito formula differs from the fundamental theorem of calculus by the additional term $\frac{1}{2}\cdot\int_0^t f''(X_s)d\langle X\rangle_s$, where the quadratic variation $\langle X\rangle_t$ is an Ito process.

Differential notation. To state Ito´s formula it is convenient to use the symbolic differential notation

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}\cdot f''(X_t)d\langle X\rangle_t$$

Multi-dimensional Ito formula

Let $X(t) = (X_1(t), ..., X_n(t))$ be an n-dimensional Ito process with

$$X_i(t) = X_i(0) + \int_0^t F_i(s)ds + \sum_{j=1}^{m}\int_0^t G_{ij}(s)dW_j(s), \; i = 1, .., n,$$

where $W(t) = (W_1(t), .., W_m(t))$ is an m-dimensional Brownian motion. Further let $h: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,2} - \textit{function}$ i.e. h is continuous, continuously differentiable with respect to the first variable (time) and twice continuously differentiable with respect to the last n variables (space). We then have

$$h(t, X_1(t), .., X_n(t) = h(0, x_1(0), ..., X_n(0)$$
$$+ \int_0^t h_t(s, X_1(s), ..., X_n(s))ds + \sum_{i=1}^n \int_0^t h_{x_i}(s, X_1(s), ..., X_n(s))dX_i(s)$$
$$+ \tfrac{1}{2}\cdot\sum_{i,j=1}^n \int_0^t h_{x_i x_j}(s, X_1(s), ..., X_n(s))d\langle X_i, X_j\rangle_s$$

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Ornstein-Uhlenbeck

The question investigated in this section is given by

$$dX_t = -\alpha X_t dt + \sigma dB_t \text{ } with \text{ } X_0 = x_0 \tag{3}$$

where α and σ are positive constants. This model became well known after 1931, when it was used by the physicists Orsntein and Uhlenbeck to study behavior of gasess. In finance this process was used in one of the first stochastic models for interest rates. In that context, X_t was intended to capture the deviation of an interest rate around a given fixed rate.
Even when this equation has a simple solution, this is not of the form $f(t, B_t)$, so the method based on Ito´s formula is useless.

Solving O-U SDE

It is possible to look for a solution to equation (3) in the larger class of processes that can be written as

$$X_t = a(t)\{x_0 + \int_0^t b(s)dB_s\}$$

applying the product rule we find that

$$dX_t = a'(t)\{x_0 + \int_0^t b(s)dB_s\}dt + a(t)b(t)dB_t$$

If we assume that $a(0) = 1$ and $a(t) > 0$ for all $t \geq 0$, then the process defined is a solution of the SDE:

$$dX_t = \frac{a'(t)}{a(t)} X_t dt + a(t)b(t)dB_t \text{ } with \text{ } X_0 = x_0 \tag{4}$$

For the coefficients of (4) to mach those of equation (3), we only need to satisfy the simple equations

$$\frac{a'(t)}{a(t)} = -\alpha \text{ } and \text{ } a(t)b(t) = \sigma$$

the solution to equation (3) is given by

$$X_t = e^{-\alpha t}\{x_0 + \sigma \int_0^t e^{\alpha s} dB_s\} = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s \tag{5}$$

Here we have that $E(X_t) = x_0 e^{-\alpha t}$, and using the Ito isometry to compute the variance of X_t, we find

$$Var(X_t) = \sigma^2 \int_0^t e^{-2\alpha(t-s)} ds = \frac{\sigma^2}{2\alpha}\{1 - e^{-2\alpha t}\} \to \frac{\sigma^2}{2\alpha}$$

Definition

Let (Ω, F, P) be a given probability space and $T \in \mathbb{R}$. Defining $\{X_t, t \in T\}$ random variables on Ω and $\{F_t, t \in T\}$ a collection of sub-σ-fields of F. Then

a) $\{F_t, t \in T\}$ is a filtration of F if the collection is not decreasing, i.e.,

$$F_s \subset F_t \forall s, t \in T, \, with \, s < t$$

b) the collection $\{X_t, t \in T\}$ is adapted to the filtration $\{F_t, t \in T\}$ if is F_t-measurable for all $t \in T$

c) $\{X_t, t \in T\}$ is a martingale with respect to $\{F_t, t \in T\}$ if

- $\{F_t, t \in T\}$ is a filtration of F
- $\{X_t, t \in T\}$ is adapted to $\{F_t, t \in T\}$
- $X_t \subset L_1 \forall t \in T$
- $E(X_t \mid F_s) = X_s \forall s, t \in T, \, with \, s \leq t$

If the last equality is replaced for $\geq$ or $\leq$ i.e., for all $s \leq t$

$$E(X_t \mid F_s) \geq X_s \, or \, E(X_t \mid F_s) \leq X_s$$

then we have that $\{X_t, F_t, t \in T\}$ is a submartingale or a supermantigale respectively.

Girsanov Theorem

General assumption for this section:

Let $\{(X(t), F_t)\}_{t\geq 0}$ be an m-dimensional progressively measurable process where $\{F_t\}$is the Brownian filtration with

$$\int_0^t X_i^2(s)ds < \infty \; a.s. \; P \; for \; all \; t \geq 0, \; i = 1, ..., m.$$

Let further

$$Z(t, X) := exp\left(-\sum_{i=1}^{m}\int_0^t X_i(s)dW_i(s) - \frac{1}{2}\int_0^t \|X(s)\|^2 \, ds\right)$$

As the argument in $Z(t, X)$ is an Ito process, the Ito formula, this implies

$$Z(t, X) = 1 - \sum_{i=1}^{m}\int_0^t Z(s, X)X_i(s)dW_i(s)$$

Thus, $Z(t, X)$ is a continuous local martingale with $Z(0, X) = 1$. As $Z(t, X)$ is also positive, it is a super-martingale. If $Z(t, x)$ is even a martingale then we have $E(Z(t, X)) = 1$ for all $t \geq 0$. Then, for all $T \geq 0$ we can define a probability measure Q_T for F_T via

$$Q_T(A) := E(1_A \cdot Z(T, X)) \; for \; A \epsilon F_T \tag{1}$$

Hence, $Z(T, X)$is the Radon-Nikodym density of Q_T with respect to P. The so-defined family of probability measures has the following consistency property

$$Q_T(A) = Q_t(A)$$

for all $A\epsilon F_t$, $t\epsilon\, [0, T]$,because we have

$$Q_T(A) = E(1_A \cdot Z(T, X)) = Q_t(A)$$

In particular, for bounded stopping times $0 \leq \tau \leq T$ and $A\epsilon F_\tau$the optional sampling theorem yields

$$Q_T(A) = E(1_A \cdot Z(T,X)) = E(E(1_A \cdot Z(T,X) \mid F_\tau)) = Q_\tau(A)$$

The following theorem now demonstrates the way a $Q_T - Brownian$ motion $W^{Q_T}(t)$ can be constructed from a P-Brownian motion $W(t)$ via a change of measure from P to Q_T.

Theorem. *Let the process $Z(t,X)$ be a martingale and define the process $\{(W^Q(t), F_t)\}_{t \geq 0}$ by*

$$W_i^Q(t) := W_i(t) + \int_0^t X_i(s)ds,\ 1 \leq i \leq m,\ t \geq 0$$

Then, for each fixed $T \epsilon [0,\infty)$ the process $\{(W^Q(t), F_t)\}_{t \epsilon [0,T]}$ is an m-dimensional Brownian motion on (Ω, F_T, Q_T) where the probability measure Q_T is defined in (1).