Lecture 5

Dept. Of Electrical Engineering

Linköping University, Sweden

Outline

- The volatility problem using BSM model
- Volatility modeling
- Models other than BSM

The volatility problem

Black-Scholes-Merton model (***BSM***)

- When stock price follows GBM, unique arbitrage free price
  - solution of a PDE (diffusion type)
  - depends on volatility, not the drift !

- Closed form solution for European options

- Writer of the options
  - receives the up-front payment (price)
  - implement a fully replicating, self financing strategy

- Fair price of derivatives : requires ‘volatility’
- *BSM*: volatility assumed to be constant (Empirical evidence?)

- Common assumption : ‘frictionless’ market
  - no transaction cost
  - same interest rate for borrowing & saving
  - bonds and stocks available as real number, in unlimited quantities

- GBM unrealistic in practice!

- Stylized facts :
  Asset price follows fat tails and skew in the probability distribution

Do the market incorporate these stylized fact in pricing options?

- **Historic volatility** :

- S_i : the *stock price* at end of day i
- *return* as $u_i = (S_i - S_{i-1})/S_{i-1}$
- Define σ_n as the volatility per day between day n-1 and day n,
- Assumption : *return* is zero mean process

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

- *Weighting*

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \quad \text{with} \quad \sum_{i=1}^{m} \alpha_i = 1$$

Historic data is rarely a good guide to the future !!

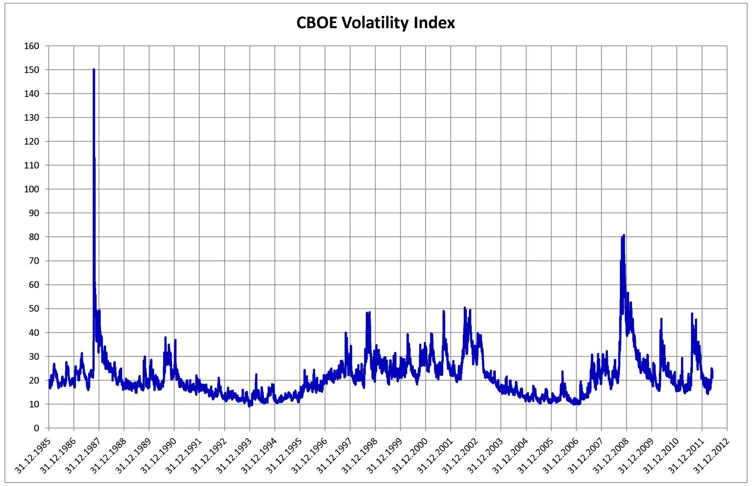
- **Implied volatility** :
  volatility obtained from *BSM* model when *BSM* implied price set to market price

  - market's view of future actual volatility over lifetime of the particular option.
  - too optimistic about perfect knowledge on future !!
  - market price not necessarily 'fair': depends on *supply & demand*
  - option price (hence, volatility) depends on *panic & greed*

VIX

Market volatility index based on implied volatility of S&P 500 index options

represents one measure of the market's expectation of stock market volatility over the next 30 day period.



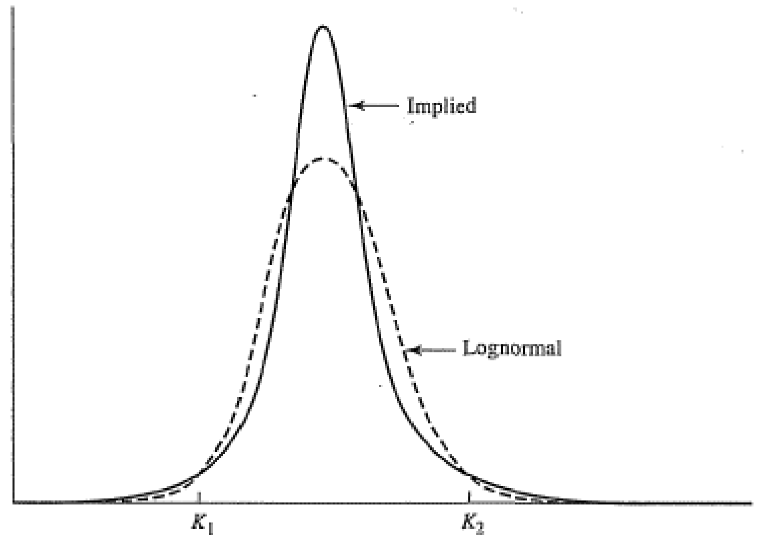
Source: Wikipedia

Practical issues with implied volatilities

- Dozens of options on any stock with different strikes and maturities
- Implied volatility can be computed for each option sold
- Implied volatilities do not always match !

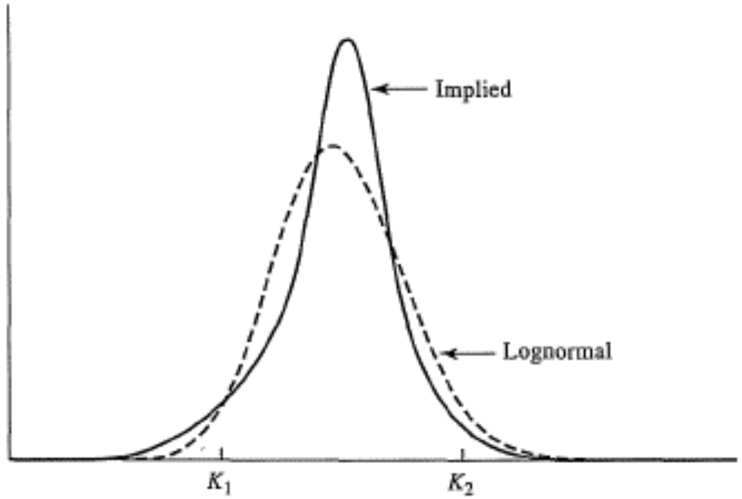
Note : implied volatilities for European put & call options (with same strike & expiry) same

Volatility Smile for FX options



Ref: J. C. Hull

- Volatility smirk for equity options



Ref: J. C. Hull

Volatility term structure

- variation of implied volatility with the time to maturity of the option
- The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low

Volatility Surface

- The implied volatility as a function of the strike price and time to maturity

Role of the model

- Not possible to perfectly hedge in a model free sense
- Translate to model risk
- Can’t aspire to the realism of physical modeling
- Can model explain stylized facts?
- Model: most effective when similar derivatives not traded actively

Black-Scholes-Merton model

- *BSM* provides a gross approximation to option price behavior
- *BSM*: interpolation tool
  - to price option consistent with other actively traded options

Modeling volatilities

Stylized facts

- Returns may have heavy tails, skewness
- Volatility clustering
- Leverage effect
- Smile/Smirk dynamics

Volatility Model

- Time series based
- Stochastic volatility
- …..

Time series based model

Given a *return* time series $R_1, R_2 \dots R_n$

Volatility (σ_n) : the <u>conditional</u> standard deviation of R_n given $R_1, R_2, \dots R_{n-1}$

return modeled as zero mean process

$\{\varepsilon_n\}$ *i.i.d.* sequence with zero mean and unit variance

$$R_n = \sigma_n \varepsilon_n$$

Model volatility as time varying (conditional heteroscedastic)

Cf.: for ARMA model, conditional mean is time varying but cond. Variance is constant

ARCH(m) Model

V_L : long-run variance rate

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i R_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

EWMA model

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1} R_{n-i+1}^2;$$

leading to $\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) R_{n-1}^2$

GARCH(1,1)

$$\sigma_n^2 = \gamma V_L + \alpha R_{n-1}^2 + (1 - \gamma - \alpha)\sigma_{n-1}^2$$

GARCH(p,q)

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{p} \alpha_i R_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

- *return* has time varying conditional variance but constant unconditional variance

$$R_n = \sigma_n \varepsilon_n$$ and ε_n is stationary sequence

- *return* sequence is uncorrelated although dependent

$$E(R_n R_{n-k}) = E\{E(R_n R_{n-k} \mid F_{n-1})\} = 0$$
$$\mathrm{var}(R_n \mid F_{n-1}) = \sigma_n^2$$

Conditional Variance depends on past value of the process !!

- *return* is a weak white noise process

- Unconditional variance for GARCH(1,1) process?

Given

$$\sigma_n^2 = \omega + \alpha R_{n-1}^2 + \beta\sigma_{n-1}^2$$

Define

$$\eta_n = R_n^2 - \sigma_n^2$$

Since $E_{n-1}(\eta_n) = E_{n-1}(R_n^2) - \sigma_n^2 = 0,$

and $\operatorname{cov}(\eta_n \eta_{n-1}) = E\ (\eta_n \eta_{n-1}) = E\{E_{n-1}(\eta_n \eta_{n-1})\}$

$= E\{\eta_{n-1} E_{n-1}(\eta_n)\} = 0;$

- weak white noise process

- Using simple algebra

$$\sigma_n^2 = \omega + (\alpha + \beta)R_{n-1}^2 - \beta\eta_{n-1}$$

So,

$$R_n^2 = \sigma_n^2 + \eta_n = \omega + (\alpha + \beta)R_{n-1}^2 - \beta\eta_{n-1} + \eta_n .$$

Assume $(\alpha + \beta) < 1$

and $\mu = \omega / \{1 - (\alpha + \beta)\},$

Then

$$(R_n^2 - \mu) = (\alpha + \beta)(R_{n-1}^2 - \mu) - \beta\eta_{n-1} + \eta_n .$$

$\Rightarrow$ R_n^2 is an ARMA (1,1) process with mean μ

$\Rightarrow$ μ is unconditional variance of R_n

- GARCH model for $\{R_n\}$ same as ARMA for $\{R_n^2\}$ with weak white noise
- ACF of $\{R_n^2\}$ is readily available !!!
- GARCH can't model *leverage*, extensions…
- Parameter estimation, model order ??

How Good is the Model ?

Standardized residuals $R_n / \hat{\sigma}_n$ estimate $\{\varepsilon_n\}$

If ACF of squared standardized residuals shows little autocorrelation, good fit

- Forecasting Future Volatility using GARCH(1,1)

$$E[\sigma_{n+k}^2] = V_L + (\alpha + \beta)^k (\sigma_n^2 - V_L)$$

The variance rate for an option expiring on day m is

$$\frac{1}{m} \sum_{k=0}^{m-1} E\left[\sigma_{n+k}^2\right]$$

The volatility per annum for an option lasting T days

$$\sqrt{252\left(V_L + \frac{1-e^{-aT}}{aT}\left[V(0)-V_L\right]\right)}$$

where

$$a = \ln\frac{1}{\alpha+\beta}$$

Stochastic volatility model

- Realised volatility of traded assets displays significant variability.
- Idea: treat (spot) volatility as random and model it
- Ex: Diffusion process

$$dS_t = \mu S_t dt + \sqrt{\upsilon_t} S_t dW_t$$

$$d\upsilon_t = \alpha_{s,t}(\upsilon_t)dt + \beta_{s,t}(\upsilon_t)dB_t$$

$$Corr(dW_t, dB_t) = \rho$$

Stochastic volatility model

- Two sources of randomness but one traded asset for hedging

  - incomplete market !
  - market price of volatility risk

- Not all risk can be dynamically hedged away
- assign a value to that risk.

Heston model

Instantaneous variance follows mean reverting process

$$dS_t = \mu S_t dt + \sqrt{\upsilon_t} S_t dW_t \qquad 2\theta\omega \succ \xi^2$$

$$d\upsilon_t = \theta(\omega - \upsilon_t)dt + \xi\sqrt{\upsilon_t} dB_t \qquad \theta \succ 0$$

(Semi) analytical solution available for pricing European options

Cf. Diffusion limit of GARCH (1,1) (ref: Nelson)

$$d\upsilon_t = \theta(\omega - \upsilon_t)dt + \xi\upsilon_t dB_t$$