# Stochastic systems

## Market price of risk, Feynman-Kac

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## Outline





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The martingale theory of arbitrage pricing is one of the greatest triumphs of probability theory since it has direct bearing on financial transactions. Here a simple calculation has some claim to being the most important in this topic. Let's simply begin with the pricing formula $V_t = \beta_t U_t$ and work our way toward an equation

$$dV_t = d(\beta_t U_t) = \beta_t dU_t + U_t d\beta_t$$
$$= \frac{u(\omega,t)}{d(\omega,t)} dS_t + \left\{ U_t - \frac{u(\omega,t)}{d(\omega,t)} D_t \right\} d\beta_t$$

This calculations gives us the required candidates for the portfolio weights:

$$a_t = \frac{u(\omega,t)}{d(\omega,t)} \text{ and } b_t = U_t - \frac{u(\omega,t)}{d(\omega,t)} D_t$$

These formulas are very important for the martingale theory of pricing.

## Outline

1. The market price of risk
   - The portfolio weights
   - The self-financing check
2. Feynman-Kac
   - The Feynman-Kac connection for Brownian motion
   - Feynman-Kac and the Black-Scholes

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## Self-financing check

By construction of the portfolio weights $a_t$ and $b_t$ we have

$$dV_t = a_t dS_t + b_t d\beta_t \text{ for } t \in [0,T]$$

if we want to show that the portfolio determined by $(a_t, b_t)$ is self-financing we only need to show that we also have

$$V_t = a_t S_t + b_t \beta_t$$

Evaluating the right-hand side

$$a_t S_t + b_t \beta_t = U_t \beta_t + \frac{u(\omega,t)}{d(\omega,t)} S_t - \frac{u(\omega,t)}{d(\omega,t)} D_t \beta_t$$

and since $\beta_t U_t = V_t$ and $D_t \beta_t = S_t$ the preceding formula simplifies to just i.e., does represent the value of a self-financing portfolio.



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Before going much further, at minimum we must be sure that we really do have a probability measure $Q$ that makes the discounted stock price $D_t = S_t/\beta_t$ into a $Q$-martingale. From Girsanov theory, the first step toward the determination of such a measure is to work out the SDE for $D_t$. An easy way is simply to apply Ito's formula to $D_t$ and turn the crank:

$$dD_t = d(S_t/\beta_t) = D_t\{(\mu_t - r_t)\,dt + \sigma_t dB_t\} \tag{1}$$

From this $D_t$ would be a local martingale if we could remove the drift term $\mu_t - r_t$. If we define the measure $Q$ by taking $Q(A) = E_P(1_A M_T)$, where $M_t$ is the exponential process

$$M_t = exp(-\int_0^t m_t dB_t - \frac{1}{2}\int_0^t m_t^2 dt) \text{ with } m_t = \{\mu_t - r_t\}/\sigma_t \tag{2}$$

Lecture 4

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If $m_t = \{\mu_t - r_t\}/\sigma_t$ is bounded, then $M_t$ is a martingale, and the process defined by

$$d\tilde{B}_t = dB_t + m_t dt$$

is a $Q$-Brownian motion. Finally, the SDE for $D_t$ given by equation (1) can be written in terms of $\tilde{B}_t$ as

$$dD_t = D_t \sigma_t d\tilde{B}_t = S_t \sigma_t / \beta_t d\tilde{B}_t$$

where $D_t$ is a $Q$-local martingale.

Lecture 4

## Market price of risk

The quantity $m_t = \{\mu_t - r_t\}/\sigma_t$ has an economic interpretation. The ratio $\{\mu_t - r_t\}/\sigma_t$ measures, in units of $\sigma_t$, the excess of the rate of return of the risky security $S_t$ over the riskless security $\beta_t$. For this reason, $m_t$ is often called the ***market price of risk***. Models for which $m_t = 0$ are called ***risk neutral models***, and by the form of the Girsanov transform (2) we see that such models have the property that $P = Q$.



## The Feynman-Kac connection

The basic Feynman-Kac representation theorem tells us that for any pair of bounded functions $q:\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ and for any bounded solution $u(t,x)$ of the individual problem

$$u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + q(x)u(t,x)\ u(0,x) = f(x) \tag{3}$$

we can represent $u(t,x)$ by the *Feynman-Kac* formula:

$$u(t,x) = E\left[f(x+B_t)exp\left(\int_0^t q(x+B_s)ds\right)\right] \tag{4}$$

The most immediate benefit of the Feynman-Kac formula (4) is that it gives us a way to get information on the global behavior of a sample path of Brownian motion.



## Outline





## Feynman-Kac representation theorem for Brownian motion

Suppose that the function $q:\mathbb{R}\to\mathbb{R}$ is bounded, and consider the initial-value problem

$$u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + q(x)u(t,x)\ u(0,x) = f(x) \tag{5}$$

where $f: R\to\mathbb{R}$ is also bounded. If $u(t,x)$ is the unique bounded solution of initial-value problem (5), then $u(t,x)$ has the representation

$$u(t,x) = E\left[f(x+B_t)exp\left(\int_0^t q(x+B_s)ds\right)\right]$$

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The traditional hunting ground for financial applications of the Feynman-Kac method is the class of stock and bond models that may be written as

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dB_t \text{ and } d\beta_t = r(t,S_t)\beta_t dt$$

where the model coefficients $\mu(t,S_t)$, $\sigma(t,S_t)$ and $r(t,S_t)$ are given by explicit functions of the current time and current stock price. This models contains at the barest minimum, the classic Black-Scholes model where the coefficients take the specific forms

$$\mu(t,S_t) \equiv \mu S_t, \sigma(t,S_t) \equiv \sigma S_t, \text{ and } r(t,S_t) \equiv r$$

for constants $\mu, \sigma$, and $r$.