Stochastic systems

Black-Scholes and Greek letters

ISY- Automatic control
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November, 2012

Outline

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Introduction

- In the early 70s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.This model has had a huge influence on the way that traders price and hedge derivatives.
- Black and Scholes used the capital asset pricing model to determine a relationship between the market´s required return on the option to the required return on the stock. This relationship depends on the stock price and time.
- Merton´s approach involved setting up a riskless portfolio consisting of the option and the underlying stock and arguing that the return on the portfolio over a short period of time must be the risk-free return.

- The Black-Scholes-Merton is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. The arguments used involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate r.
- This can be set up since the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Assumptions

- The short selling of securities with full use of proceeds is permitted.
- There are no transactions costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of the derivative.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.
- The risk-free rate of interest, r, is constant and the same for all maturities.
- The stock-price follows the process $dS = \mu S dt + \sigma S dz$ with σ and μ constant.

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Let's consider a derivative's price at a general time t (not at time zero). If T is the maturity date, the time to maturity $T - t$. Consider the stock price process

$$dS = \mu S dt + \sigma S dz \tag{1}$$

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function on S and t. Applying the Ito formula we get

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz \tag{2}$$

The discrete versions of the previous equations are

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \tag{3}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z \tag{4}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. A portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated. Define Π as the value of the portfolio $\Pi = -f + \frac{\partial f}{\partial S} S$. The change $\Delta \Pi$ in the value of the portfolio Δt in the time interval is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \partial S \tag{5}$$

Replacing (3) and (4) into (5) yields

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{6}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. It follows that

$$\Delta \Pi = r \Pi \Delta t \tag{7}$$

where r is the risk-free interest rate. Substituting Π and (6) into (7), we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left(f - \frac{\partial f}{\partial S} S \right) \Delta t \tag{8}$$

So that

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{9}$$

This equation is known as the Black-Scholes-Merton differential equation.

- Any function $f(S,t)$ that is a solution of this equation is the theoretical price of a derivative that could be traded.
- If a function $f(S,t)$ does not satisfy this differential equation, it cannot be the price of a derivative without creating arbitrage opportunities for traders.

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Nature of the solution

The only special function needed to represent the solution is the Gaussian integral Φ, and all of the parameters of our model appear in a thoroughly intuitive way. Specifically, the arbitrage price of the European call option at time t with a current stock price of S, exercise time T, strike price K, and residual time $\tau = T - t$ is given by

$$S\Phi\left(\frac{log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \qquad (10)$$

Perhaps the most important practical feature of the B-S formula is that tells us, simply and exactly, how to build the replicating portfolio. this formula not only specifies the arbitrage price, it even tells how to enforce that price.

Black-Scholes via Martingales

Among the cases where the martingale valuation formula may be worked out explicitly there's the European call under Black-Scholes model. Here the basic task is to calculate the time-zero value

$$V_0 = e^{-rT} E_Q\left[(S_T - K)_+\right] \qquad (11)$$

There are two basic steps to organize this calculation.

- We must understand the measure Q that makes $Dt = S_t/\beta_t$ a martingale.
- Then we must exploit that understanding to calculate the expectation in equation (11).

There's always the option of writing Q in terms of P and an exponential martingale. In most cases we can get what we need more quickly just by using the fact that we know the distribution of $\{S_t\}$ under Q.

In the case of the classical economy specified by (1), we can simply solve the stock bond SDE to find an explicit representation of the process D_t as

$$S_t/\beta_t = S_0 exp\left(\left(\mu - r - \sigma^2/2\right)t + \sigma B_t\right) \qquad (12)$$

Then we can apply Ito's formula to the representation (12) to find

$$d(S_t/\beta_t) = \sigma(S_t/\beta_t)\{d(t(\mu - r)/\sigma + B_t)\} \qquad (13)$$

This SDE tells us that D_t will be a Q-local martingale provided that Q is chosen so that the process defined by

$$\tilde{B} = \frac{\mu - r}{\sigma}t + B_t$$

is a Q-Brownian motion.

The defining property of Q is that $\tilde{B}_t$ is a Q-Brownian motion, so the most logical way to compute the expectation in (11) is to express S_t in terms of $\tilde{B}_t$. We can write S_T as a function of B_T, and this as a function of $\tilde{B}$, so after this 2 steps, we find

$$S_T = S_0 exp(rT - \sigma^2 T/2 + \sigma\tilde{B}_T)$$

The time-zero valuation formula for a European call option with strike price K can then be written as

$$V_0 = e^{-rT} E_Q\left[\left(S_0 exp(rT - \sigma^2 T/2 + \sigma\tilde{B}_T) - K\right)_+\right]$$

If $Y = -\sigma^2 T/2 + \sigma\tilde{B}_T$ is a Gaussian variable with mean $-\sigma^2 T/2$ and variance $\sigma^2 T$, then V_0 equals the expectation of a function of Y that we may write explicitly as an integral

$$S_0 \int_{log(K/S_0)-rT}^{\infty} e^y e^{-(y+\sigma^2 T/2)^2/2\sigma^2 T} \frac{dy}{\sigma\sqrt{2\pi T}}$$

$$-e^{-rT} K \int_{log(K/S_0)-rT}^{\infty} e^{-(y+\sigma^2 T/2)^2/2\sigma^2 T} \frac{dy}{\sigma\sqrt{2\pi T}}$$

The sum of these integrals may be reduced to $f(0, S_0)$, where $f(t,x)$ is the solution of the Black-Scholes that is given by (10) where we use τ as the usual shorthand for $T-t$.

Feynman-Kac and the Black-Scholes

The traditional hunting ground for financial applications of the Feynman-Kac method is the class of stock and bond models that may be written as

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dB_t \text{ and } d\beta_t = r(t,S_t)\beta_t dt \tag{14}$$

where the coefficients take the specific forms

$$\mu(t,S_t) \equiv \mu S_t,\ \sigma(t,S_t) \equiv \sigma S_t,\ r(t, S \equiv r) \tag{15}$$

if we simply repeat the original derivation of the classic Black-Scholes PDE, we can show that the time t arbitrage price $u(t,S_t)$ of the time T European claim $X = h(S_T)$ will satisfy the terminal-value problem

$$u_t(t,x) = -\frac{1}{2}\sigma^2(t,x)u_{xx}(t,x) - r(t,x)xu_x(t,x) + r(t,x)u(t,x) \tag{16}$$

$$u(T,x) = h(x) \tag{17}$$

We will assume that the nonnegative interest rate process $r(t,x)$ is bounded, and that the stock parameters $\mu(t,x)$ and $\sigma(t,x)$ satisfy both the Lipschitz condition and the linear growth rate condition, and even assume that $h(x)$ is bounded.
If we replace $h(x)$ by $h_0(x) = min(h(x), M)$, where M denotes the total of all of the money in the universe, then h_0 is bounded. With these ground rules, we are ready to state the Feynman-Kac Formula for the solution of the general Black-Scholes PDE.

If $u(t,x)$ is the unique bounded solution of the terminal-value problem given by equations (16) and (17), then $u(t,x)$ has the representation

$$u(t,x) = E\left[h(X_T^{t,x})exp(-\int_t^T r(s, X_s^{t,x} ds)\right]$$

where for $s \in [0,t]$ the process $X_s^{t,x}$ is defined by taking $X_s^{t,x} \equiv x$ and where for $s \in [t,T]$ the process $X_s^{t,x}$ is defined to be the solution of the SDE:

$$dX_s^{t,x} = r(s, X_s^{t,x})X_s^{t,x} dt + \sigma(s, X_s^{t,x})dB_s \text{ and } X_t^{t,x} = x$$

Introduction

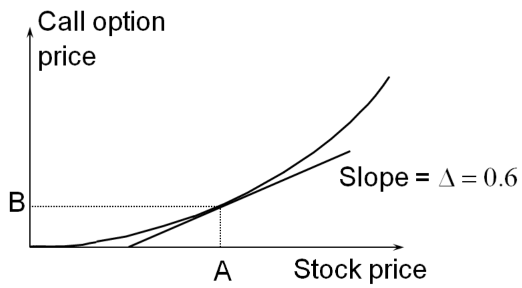
Definition

The Greek letters or "Greeks" measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable.

Outline

Delta Hedging

The delta (Δ) of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price.



When the stock price correspond to point A, the option price corresponds to point B, and Δ is the slope of the line indicated. In general

$$\Delta = \frac{\partial c}{\partial S}$$

where c is the price of the call option and S is the stock price.

Delta of European Stock Options. For a European call option on a non-dividend-paying stock, delta is given by

$$\Delta(put) = N(d_1) - 1$$

Delta is negative, which means that a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock.

Delta of a Portfolio. The delta of a portfolio of options or other derivatives dependent on a single asset whose price is S is

$$\frac{\partial \Pi}{\partial S}$$

where Π is the value of the portfolio.
This can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity ω_i of option $i(1 \le i \le n)$, the delta of the portfolio is given by

$$\Delta = \Sigma_{i=1}^{n} \omega_i \Delta_i$$

where Δ_i is the delta of the *ith* option. The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When the position has been taken, the portfolio is referred to as being *delta neutral*.

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Theta

The theta of a portfolio (Θ) of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same, and is also known as the time decay. The formula for a European call option on a non-dividend-paying stock is given by

$$\Theta(call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where and are defined by the B-S pricing formulas and

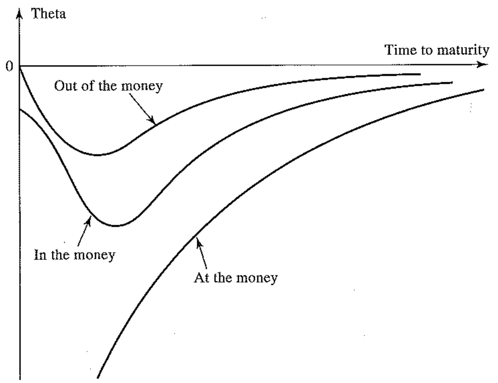
$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{18}$$

is the probability density function for a standard normal distribution.

For a European put option on the stock,

$$\Theta(put) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Because $N(-d_2) = 1 - N(d_2)$, the theta of a put exceeds the theta of the corresponding call by rKe^{-rT}. Here T=years, when theta is quoted, time measure is days. We can measure theta either “per calendar day” or “per trading day”.



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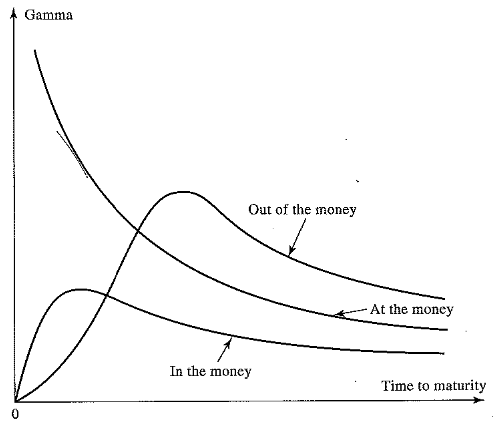
Gamma

The gamma (Γ) is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If $\Gamma$is small Δchanges slowly, and adjustment to keep a portfolio Δ neutral need to be made only relatively infrequently. However, if Γ is highly negative or highly positive, $\Delta$is very sensitive to the price of the underlying asset.

For a European call or put option on a non-dividend-paying stock, we have $\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$ where d_1 is defined by the B-S pricing formulas and as in (18).



Outline

The Black-Scholes-Merton model
The Greek letters
Delta Hedging
Theta
Gamma
Vega
Rho

Vega

The vega of a portfolio derivatives, $\mathcal{V}$, is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$$

If $\mathcal{V}$ is highly positive or highly negative, the portfolio's value is very sensitive to small changes in volatility. If it is close to zero, volatility changes have relatively little impact on the value of the portfolio. For a European call or put on a non-dividend-paying stock, vega is given by

$$\mathcal{V} = S_0 \sqrt{T} N'(d_1)$$

where d_1 is defined by the B-S pricing formulas and as in (18).

Lecture 5

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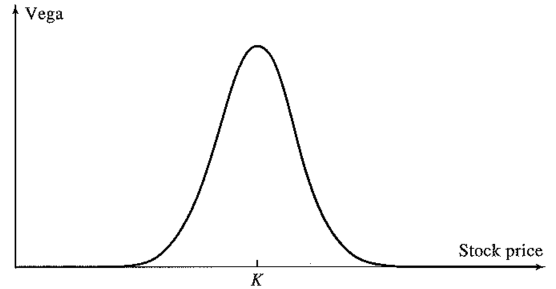
The vega of a long position in a European or American option is always positive.



Lecture 5

The Black-Scholes-Merton model
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Outline

1. The Black-Scholes-Merton model
   - Introduction
   - Derivation of the Black-Scholes-Merton differential equation.
   - Solution
2. The Greek letters
   - Delta Hedging
   - Theta
   - Gamma
   - Vega
   - Rho

Lecture 5

Rho

The rho of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate:

$$\frac{\partial \Pi}{\partial r}$$

It measures the sensitivity of the value of a portfolio to a change in the interest rate when all else remains the same. For a European call option on a non-dividend-paying stock,

$$rho\,(call) = KTe^{-rT}N(d_2)$$

where d_2 is defined by the B-S pricing formulas. For a European put option,

$$rho\,(put) = -KTe^{-rT}N(-d_2)$$