Exercises

- Assume that the price of an underlying at a time t, S_t, is described by

$$dS_t = \mu(S_t, t)\, dt + \sigma(S_t, t)\, dW_t$$

where $(W_t)_{t\epsilon[o,T]}$ is a standard Brownian motion defined on a probability space with an augmented filtration $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\epsilon[0,T]}, P\right)$. Applying Ito lemma to $c = c(S_t, t)$ we get

$$dc = \left(\frac{\partial c}{\partial t} + \mu(S_t, t)\frac{\partial c}{\partial S_t} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2}{\partial S_t^2}\right) dt + \frac{\partial c}{\partial S_t}\sigma(S_t, t)\, dW_t$$

or in an equivalent way

$$dc = \mu_c(S_t, t)\, cdt + \sigma_c(S_t, t)\, cdW_t$$

where

$$\mu_c(S_t, t) = \left(\frac{\partial c}{\partial t} + \mu(S_t, t)\frac{\partial c}{\partial S_t} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 c}{\partial S_t^2}\right)/c$$

and $\sigma_c(S_t, t) = \left(\frac{\partial c}{\partial S_t}\right)\frac{\sigma(S_t,t)}{c}$

Consider now a portfolio with value $\Pi_t = \alpha S_t + \beta_c$. Using arbitrage arguments show that

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 c}{\partial S_t^2} + rS_t\frac{\partial c}{\partial S_t} - rc = 0$$

- Assume that the Black-Scholes model holds and consider the case of a bank that has written (sold) a call option on the stock S with the parameters $S(0) = S_0 = 760$, $r = 0.06$, $\sigma = 0.65$, $K = S_0$,with an exercise date, $T = 1/4$ years. Using the exact solution of the Black Scholes call price, computed by the Matlab code

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% Black-Scholes call option computation function y = bsch(S,T,K,r,sigma);
normal = inline('(1+erf(x/sqrt(2)))/2','x');
d1 = (log(S/K)+(r+.5*sigma^2)*T)/sigma/sqrt(T);
d2 = (log(S/K)+(r-.5*sigma^2)*T)/sigma/sqrt(T);
y = S*normal(d1)-K*exp(-r*T)*normal(d2);
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Choose a number of hedging dates, N, and time steps . Assume that $\beta(0) = -f_S(0, S_0)$ and then

- Write a code that computes the $\Delta \equiv \partial f(0, S_0)/\partial S_0$ of a call option

-Generate a realization for $S(n\Delta t, \omega), n = 0, ..., N$.

- Generate the corresponding time discrete realizations for the processes $\alpha_n$and $\beta_n$and the portfolio value, $\alpha_n S_n + \beta_n B_n$.

- Generate the value after settling the contract at time T,

$$\alpha_N S_N + \beta_N B_N - max\,(S_N - K, 0)$$

Compute with only one realization and several values of N. How would you proceed if you don't have the exact solution of the Black-Scholes equation?

- Assume the Black-Scholes world and consider a standard european call struck at the money with 12 months to maturity. If interest rate $r = 0.04$, is $\delta >$ or < 0.5 ? (Hint: use δ for Black-Scholes and consider the properties of standard normal distribution)

- Consider the following stationary model:

$$\begin{aligned} x_t &= \sqrt(v_t)\epsilon_t \\ y_t &= \sqrt(v_t)(\epsilon_t + \eta_t) \\ v_t &= 1 + \epsilon_{t-1}^2 + 2\eta_{t-1}^2 + 0.25v_{t-1}, \end{aligned} \tag{1}$$

  where η_t and ϵ_t are i.i.d. sequences with zero mean and unit variance. Compute covariance and correlation between x_t and y_t.

- Consider the following GARCH(1,1) model:

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{2}$$

  where $\omega = 10^{-5}$,$\alpha = 0.15$ and $\beta = 0.9$. Simulate the above GARCH model for $t = 1 : 1000$ and then estimate the parameters by any method of your choice. Repeat the estimation 20 times and check for possible bias. Clearly mention any assumption.

- Download the data of exchange rate of Deutsche mark per U.S. dollar from the Exercise link ($FX_DM2USD.xls$)in the course web page. Now (a) plot the log return, (b) empirical distribution of the data(e.g., using kernel density estimator). Compute the mean and variance of this distribution and plot the Gaussian distribution/density with the same mean and variance (on the same figure). Now assuming that it follows a GARCH(1,1) model, estimate the GARCH parameters and the realized volatility. Provide (any) model validation check. Write your comment based on the observation. Please specify any assumption made.

The deadline for this exercises is January 15th at 23.00.