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Anders Hultgren
15/05/00

Bertsekas: D*ynamic programming and optimal control*
Chap. 4 - Problems with perfect state information

Contents:

- Linear systems and quadratic cost,
  (*presentation with the DP-costume, some proofs)*
- Optimal stopping problems,
  (*presentation, an example)*
- Dynamic Portfolio Analysis,
  (*presentation)*
- Inventory control, (*left)*

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Linear discrete systems and quadratic cost

The system: $x_{k+1} = Ax_k + Bu_k + w_k$

Cost function: $E\left\{x_N^T Q_N x_N + \sum_{k=0}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right)\right\}$

The DP-algorithm:

$$J_N(x_N) = x_N^T Q_N x_N = x_N^T K_N x_N$$

$$J_k(x_k) = \min_{u_k} E\left\{x_k^T Q x_k + u_k^T R u_k + J_{k+1}(x_{k+1})\right\} =$$

$$= \min_{u_k} E\left\{x_k^T Q x_k + u_k^T R u_k + J_{k+1}(Ax_k + Bu_k + w_k)\right\}$$

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DP- algorithm derivations I

Next last term:

$$J_{N-1}(x_{N-1}) = \min_{u_{N-1}} E\left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + \overbrace{x_N^T K_N x_N}^{J_N} \right\}$$

$$= \min_{u_{N-1}} E\left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + (Ax_{N-1} + Bu_{N-1} + w_{N-1})^T K_N (Ax_{N-1} + Bu_{N-1} + w_{N-1}) \right\} =$$

$$= \min_{u_{N-1}} \left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + (Ax_{N-1} + Bu_{N-1})^T K_N (Ax_{N-1} + Bu_{N-1}) + E\left(w_{N-1}^T K_N w_{N-1} \right) \right\}$$

$$\tilde{J}_{N-1}(x) = \min_u \left\{ x^T \underbrace{\left(Q + A^T K_N A\right)}_{Q_{xx}} x + x^T \underbrace{A^T K_N B}_{Q_{xu}} u + u^T \underbrace{B^T K_N A}_{Q_{xu}^T} x + u^T \underbrace{\left(R + B^T K_N B\right)}_{Q_{uu}} u \right\}$$

Noise is independent of the states and the control signal and has mean value = 0

$$J(x,u) = x^T Q_{xx} x + x^T Q_{xu} u + u^T Q_{xu}^T x + u^T Q_{uu} u$$

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DP- algorithm derivations II

Minimise:

$$J(x,u) = u^T Q_{uu} u + x^T Q_{xu} u + u^T Q_{xu}^T x + x^T Q_{xx} x =$$

$$= \left(u + Q_{uu}^{-1} Q_{xu}^T x\right)^T Q_{uu} \left(u + Q_{uu}^{-1} Q_{xu}^T x\right) - x^T \left(Q_{uu}^{-1} Q_{xu}^T\right)^T Q_{uu} \left(Q_{uu}^{-1} Q_{xu}^T\right) x + x^T Q_{xx} x$$

$$u^* = -Q_{uu}^{-1} Q_{xu}^T x$$

With index and noise:

$$J_{N-1}(x_{N-1}) = x_{N-1}^T K_{N-1} x_{N-1} + E\left\{ w_{N-1}^T K_N w_{N-1} \right\}$$

$$J_0(x_0) = x_0^T K_0 x_0 + \sum_{i=0}^{N-1} E\left\{ w_i^T K_{i+1} w_i \right\}$$

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DP- algorithm derivations III

$K_{N-1} = f(K_N)$: Riccati-equation

$$K_{N-1} = Q_{xx} - Q_{xu}Q_{uu}^{-1}Q_{xu}^T = Q + A^T K_N A - A^T K_N B\left(R + B^T K_N B\right)^{-1} B^T K_N A$$

$$= A^T \left(K_N - K_N B\left(B^T K_N B + R\right)^{-1} B^T K_N\right)A + Q$$

Gives the losses and the control:

$$J_0(x_0) = x_0^T K_0 x_0 + \sum_{i=0}^{N-1} E\left\{w_i^T K_{i+1} w_i\right\}$$

$$u_k^* = -\left(R + B^T K_{k+1} B\right)^{-1} B^T K_{k+1} A x_k$$

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DP- algorithm derivations IV

$$K_{N-1} = A^T \left(K_N - K_N B\left(B^T K_N B + R\right)^{-1} B^T K_N\right)A + Q$$

Index change: $P_k = K_{N-k}$

$$P_{k+1} = A^T \left(P_k - P_k B\left(B^T P_k B + R\right)^{-1} B^T P_k\right)A + Q$$

$x_N^T K_N x_N \Leftrightarrow x_N^T P_0 x_N$ final step loss

$x_0^T K_0 x_0 \Leftrightarrow x_0^T P_N x_0$ total loss

$x_0^T P_k x_0$: Gives the loss from state x_0 in a k-step control

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Proposition 4.1

$$P_{k+1} = A^T\left(P_k - P_k B\left(B^T P_k B + R\right)^{-1} B^T P_k\right)A + Q$$

(A, B) controllable (A, C) observable, $Q = C^T C$

(a) $\exists P \geq 0 : \forall P_0 \geq 0 \Rightarrow \lim_{k\to\infty} P_k = P$

where P is the unique solution of

$$P = A^T\left(K - PB\left(B^T PB + R\right)^{-1} B^T K\right)A + Q$$

within the class of pos. semidef. sym. matrices

(b) The closed system is stable

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Convergence of the Riccati equation I

Proof:

I) Convergence of P_k in the case $P_0=0$, $P_k(0)$:
No final loss

Given the system:

$$x_{i+1} = Ax_i + Bu_i\ ,\ i = 0,1,\ldots,k-1$$

(The noise does not effect the L and the P_k)

Minimize:

$$\sum_{i=0}^{k-1}\left(x_i^T Q x_i + u_i^T R u_i\right)$$

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Convergence of the Riccati equation II

For any control sequence:

$$x_0^T P_k(0) x_0 = \min_{u_i} \sum_{i=0}^{k-1} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

$$\leq \min_{u_i} \sum_{i=0}^{k} \left(x_i^T Q x_i + u_i^T R u_i \right) = x_0^T P_{k+1}(0) x_0$$

Giving: $x_0^T P_k(0) x_0$ is monotonically increasing

Controllability gives: $x_0^T P_k(0) x_0$ is bounded from above

Then the sequence $\{P_k(0)\}$ converges,

$\lim_{k \to \infty} P_k(0) = P$ (can be shown element by element)

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Stability I

Proof:
II) Stability
For : $k \to \infty$

$$x_0^T P x_0 = \sum_{i=0}^{\infty} \left(x_i^T Q x_i + u_i^T R u_i \right) = x_0^T Q x_0 + u_0^T R u_0 + \sum_{i=1}^{\infty} \left(x_1^T Q x_1 + u_1^T R u_1 \right) =$$

$$= x_0^T \left(Q + L^T R L \right) x_0 + x_1^T P x_1$$

Gives:

$P = Q + L^T R L + D^T P D$, where $D = A + BL$

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Stability II

For any k:

$$x_k^T P x_k - x_k^T \left(Q + L^T R L\right) x_k = x_k^T D^T P D x_k$$
$$x_k^T P x_k - x_k^T \left(Q + L^T R L\right) x_k = x_{k+1}^T P x_{k+1}$$
$$x_0^T P x_0 - \sum_{i=0}^{k} x_i^T \left(Q + L^T R L\right) x_i = x_{k+1}^T P x_{k+1} \qquad (I)$$

$0 \le RHS < x_0^T P x_0$ and with $R>0$ and $Q=C^T C$ we get:

$\lim_{k\to\infty} C x_k = 0$ and $\lim_{k\to\infty} L x_k$

The observability assumption gives: $\lim_{k\to\infty} x_k = 0$

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Continuation of prop 4.1

Proof:

III) Positive definitness of P

IV) Arbitrary initial matrix P_0

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Random systems I

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

$$J_k(x_k) = \min_{u_k} \underset{w_k, A_k, B_k}{E} \left\{ x_k^T Q x_k + u_k^T R u_k + J_{k+1}(x_{k+1}) \right\} =$$

$$= \min_{u_k} \underset{w_k, A_k, B_k}{E} \left\{ x_k^T Q x_k + u_k^T R u_k + J_{k+1}(A x_k + B u_k + w_k) \right\}$$

$$K_{N-1} = E\left\{A^T K_N A\right\} - E\left\{A^T K_N B\right\}\left(R + E\left\{B^T K_N B\right\}\right)^{-1} E\left\{B^T K_N A\right\} + Q_{N-1}$$

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Random systems II

$$P_{k+1} = \tilde{F}(P_k) = \frac{E\left\{A^2\right\} P_k R}{E\left\{B^2\right\} P_k + R} + Q + \frac{T P_k^{\,2}}{E\left\{B^2\right\} P_k + R}$$

$$T = E\left\{A^2\right\} E\left\{B^2\right\} - E^2\{A\} E^2\{B\}$$

Uncertainty threshold principle:

$T > T_{divergence} \Leftrightarrow$ Riccati equation does not converge

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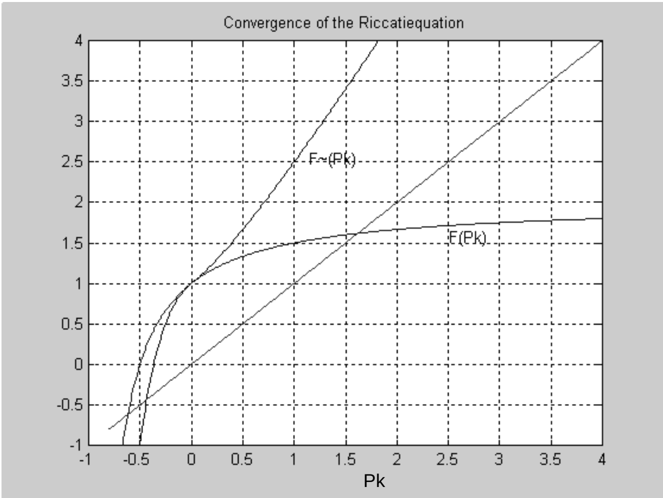
Random systems III



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Optimal stopping problem I

Asset selling: w (random offer), $\mathrm{u} = \{u^1 \quad u^2\}$ (sell, do not sell)
r (revenue)

Given system:

$$x_{k+1} = \begin{cases} T \text{ if } x_k = \mathrm{T}, \text{ or if } x_k \neq \mathrm{T} \text{ and } u_k = \mathrm{u}^1 \text{ (sell)}, \\ w_k \quad \text{otherwise} \end{cases}$$

$$\underset{\substack{w_k \\ k=0,1,\ldots,N-1}}{E} \left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k w_k) \right\}$$

$$g_N(x_N) = \begin{cases} x_N & \text{if } x_N \neq T, \\ 0 \text{ otherwise}, \end{cases} \qquad g_k(x_k, u_k w_k) = \begin{cases} (1+r)^{N-k} x_k, \text{ if } x_k \neq \mathrm{T} \text{ and } \mathrm{u_k} \neq \mathrm{u}^1 \text{ (sell)} \\ 0 \quad \text{otherwise} \end{cases}$$

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Optimal stopping problem II

The DP-algorithm:

$$J_N(x_N) = \begin{cases} x_N & \text{if } x_N \neq T, \\ 0 & \text{otherwise,} \end{cases}$$

$$J_k(x_k) \begin{cases} \max\left[(1+r)^{N-k} x_k, E\{J_{k+1}(w_k)\}\right] & \text{if } x_k \neq \mathrm{T}, \\ 0 & \text{otherwise} \end{cases}$$

Define:

$$\alpha_k = \frac{E\{J_{k+1}(w_k)\}}{(1+r)^{N-k}}$$

$x_k < \alpha_k$ do not sell

$x_k > \alpha_k$ sell

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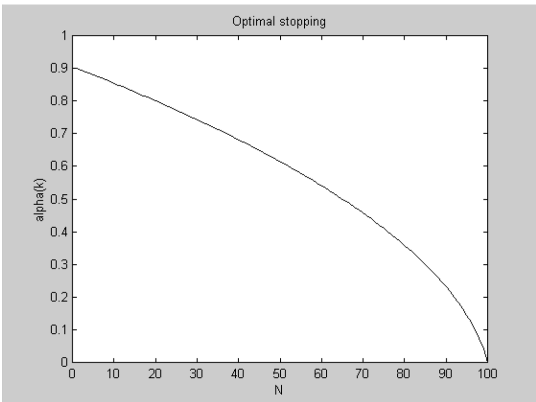
Optimal stopping problem α



Can be shown:

$\alpha_k \geq \alpha_{k+1}$, for all k

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Optimal stopping problem example

$r = 0,\ w$ - constant distribution : $\begin{matrix} p(w=0)=0.5 \\ p(w=0.5)=0.25 \\ p(w=1)=0.25 \end{matrix}$

$$J_N(x_N) = x_N = w$$

$$J_{N-1}(x_{N-1}) = \max\left[x_{N-1},\ \ E\{J_N(x_N)\}\right]$$

$$= \max\left[w,\ \ E\{w\}\right]$$

$$= \max\left[w,\ \ \frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{4}\right]$$

$$= \max\left[w,\ \ \frac{3}{8}\right]$$

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Optimal stopping problem example cont.

$$J_{N-2}(x_{N-2}) = \max\left[x_{N-2},\ \ E\{J_{N-1}(x_{N-1})\}\right]$$

$$= \max\left[w,\ \ E\left\{\max\left[w,\ \ \frac{3}{8}\right]\right\}\right]$$

$$= \max\left[w,\ \ \frac{3}{8}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{4}\right]$$

$$= \max\left[w,\ \ \frac{9}{16}\right]$$

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More optimal stopping problems

Asset buying: *w* (random price), $\mathrm{u} = \{u^1 \quad u^2\}$ (buy, do not buy)

Correlated prices: include a model

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Dynamic Portfolio Analysis I

x_0 - initial wealth
n – risky assets
e_i – random rates of return, $i=1,2,\ldots,n$.
s – rate of return, riskless asset

Wealth difference equation:

$$x_1 = s(x_0 - u_1 - \ldots - u_n) + \sum_{i=1}^{n} e_i u_i = s x_0 + \sum_{i=1}^{n} (e_i - s) u_i$$

$$\max_{u_1,\ldots,u_n} \left[E\{U(x_1)\} \right]$$

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Dynamic Portfolio Analysis II

If $\quad -\dfrac{U'(x)}{U''(x)} = a + bx$

then $\quad u^i(x) = \alpha^i (a + bsx)$

Admissible $\quad U(x):$

exponential : $e^{-\frac{x}{a}}, \quad b = 0$

logarithmic : $\ln(x + a), \quad b = 1$

power : $(1/(b-1))(a + bx)^{1-(1/b)}$, b ≠ 0, b ≠ 1